AN ℓ^1-TV ALGORITHM FOR DECONVOLUTION WITH SALT AND PEPPER NOISE

*Brendt Wohlberg**

T-7 Mathematical Modeling and Analysis
Los Alamos National Laboratory
Los Alamos, NM 87545, USA

Paul Rodríguez

Digital Signal Processing Group
Pontificia Universidad Católica del Perú
Lima, Perú

ABSTRACT

There has recently been considerable interest in applying Total Variation regularization with an ℓ^1 data fidelity term to the denoising of images subject to salt and pepper noise, but the extension of this formulation to more general problems, such as deconvolution, has received little attention. We consider this problem, comparing the performance of ℓ^1-TV deconvolution, computed via our Iteratively Reweighted Norm algorithm, with an alternative variational approach based on Mumford-Shah regularization. The ℓ^1-TV deconvolution method is found to have a significant advantage in reconstruction quality, with comparable computational cost.

***Index Terms*—** Image restoration, Deconvolution, Impulse noise, Total Variation

1. INTRODUCTION

The standard Total Variation (TV) regularization functional [1], which we shall refer to as ℓ^2-TV, may be written as

$$T(\mathbf{u}) = \frac{1}{2}\left\|K\mathbf{u} - \mathbf{s}\right\|_2^2 + \lambda \left\|\sqrt{(D_x\mathbf{u})^2 + (D_y\mathbf{u})^2}\right\|_1 ,$$

where $\mathbf{s}$ is the data, K is the linear operator representing the forward problem, and we employ the following notation:

- the p-norm of vector $\mathbf{u}$ is denoted by $\|\mathbf{u}\|_p$,
- scalar operations applied to a vector are considered to be applied element-wise, so that, for example, $\mathbf{u} = \mathbf{v}^2 \Rightarrow u_k = v_k^2$ and $\mathbf{u} = \sqrt{\mathbf{v}} \Rightarrow u_k = \sqrt{v_k}$, and
- horizontal and vertical discrete derivative operators are denoted by D_x and D_y respectively.

This functional has been applied to a wide variety of image restoration problems, including denoising [1] and deconvolution [2, 3] of images subject to Gaussian white noise.

More recently, the ℓ^1-TV functional [4, 5]

$$T(\mathbf{u}) = \left\|K\mathbf{u} - \mathbf{s}\right\|_1 + \lambda \left\|\sqrt{(D_x\mathbf{u})^2 + (D_y\mathbf{u})^2}\right\|_1$$

has attracted attention due to a number of advantages [6], including superior performance with non-Gaussian noise such as salt and pepper noise. While rapid progress has been made on the development of efficient algorithms for minimizing this functional [7, 8, 9], the majority of these methods are restricted to the denoising problem, corresponding to setting K to the identity operator, and, presumably for this reason, application of the ℓ^1-TV functional for more general inverse problems, such as deconvolution, has received little attention in the literature. In this paper we provide new experimental results for our Iteratively Reweighted Norm (IRN) algorithm [10, 11] applied to the problem of deconvolution subject to salt and pepper noise, and compare performance on this problem with that of a prominent recent approach [12] based on a different variational principle.

2. ITERATIVELY REWEIGHTED NORM APPROACH

The IRN algorithm [10, 11] for minimizing the generalized TV functional

$$T(\mathbf{u}) = \frac{1}{p}\left\|K\mathbf{u} - \mathbf{s}\right\|_p^p + \frac{\lambda}{q} \left\|\sqrt{(D_x\mathbf{u})^2 + (D_y\mathbf{u})^2}\right\|_q^q \quad (1)$$

is motivated by the Iteratively Reweighted Least Squares (IRLS) method [13, 14, 15] for solving the minimum ℓ^p norm problem $\min_{\mathbf{u}} \frac{1}{p}\|K\mathbf{u} - \mathbf{s}\|_p^p$ by solving a sequence of minimum weighted ℓ^2 norm problems. These methods represent the ℓ^p norm of $\mathbf{u}$

$$\frac{1}{p}\|\mathbf{u}\|_p^p = \frac{1}{p}\sum_k |u_k|^p,$$

by the weighted ℓ^2 norm of $\mathbf{u}$

$$\frac{1}{2}\left\|W^{1/2}\mathbf{u}\right\|_2^2 = \frac{1}{2}\mathbf{u}^T W \mathbf{u} = \frac{1}{2}\sum_k w_k u_k^2$$

with diagonal weight matrix $W = (2/p)\,\mathrm{diag}\left(|\mathbf{u}|^{p-2}\right)$. At each iteration of an iterative scheme, the ℓ^p norm is approximated by the weighted ℓ^2 norm using the weights from the previous iteration.

*This research was supported by the U.S. Department of Energy through the LANL/LDRD Program.

The weighted ℓ^2 equivalent of (1) may be written (the reader is referred to [10, 11] for full details of the derivation) as

$$T(\mathbf{u}) = \frac{1}{2}\left\|W_F^{1/2}\left(K\mathbf{u}-\mathbf{s}\right)\right\|_2^2 + \frac{\lambda}{2}\left\|\tilde{W_R}^{1/2} D\mathbf{u}\right\|_2^2 \quad ,$$

where

$$\begin{aligned} W_F &= \operatorname{diag}\left(\frac{2}{p} f_F(K\mathbf{u}-\mathbf{s})\right) \\ W_R &= \operatorname{diag}\left(\frac{2}{q} f_R\left((D_x\mathbf{u})^2 + (D_y\mathbf{v})^2\right)\right) \end{aligned}$$

$$D = \begin{pmatrix} D_x \\ D_y \end{pmatrix} \quad \tilde{W}_R = \begin{pmatrix} W_R & 0 \\ 0 & W_R \end{pmatrix} \quad ,$$

and functions (with corresponding threshold parameters ϵ_F and ϵ_R)

$$f_F(x) = \begin{cases} |x|^{p-2} & \text{if } |x| > \epsilon_F \\ \epsilon_F^{p-2} & \text{if } |x| \le \epsilon_F, \end{cases}$$

and

$$f_R(x) = \begin{cases} |x|^{(q-2)/2} & \text{if } |x| > \epsilon_R \\ 0 & \text{if } |x| \le \epsilon_R, \end{cases}$$

are required to avoid the possibility of infinite weights when $p < 2$ and $q < 2$ respectively. The minimum of this functional is

$$\mathbf{u} = \left(K^T W_F K + \lambda D^T \tilde{W}_R D\right)^{-1} K^T W_F \mathbf{s}, \qquad (2)$$

and the resulting algorithm consists of the following steps:

Initialize

$$\mathbf{u}_0 = \left(K^T K + \lambda D^T D\right)^{-1} K^T \mathbf{s}$$

Iterate

$$\begin{aligned} W_{F,k} &= \operatorname{diag}\left(\frac{2}{p} f_F(K\mathbf{u}_{k-1}-\mathbf{s})\right) \\ W_{R,k} &= \operatorname{diag}\left(\frac{2}{q} f_R\left((D_x\mathbf{u}_{k-1})^2 + (D_y\mathbf{u}_{k-1})^2\right)\right) \\ \mathbf{u}_k &= \left(K^T W_{F,k} K + \lambda D_x^T W_{R,k} D_x\right. \\ &\quad \left. + \lambda D_y^T W_{R,k} D_y\right)^{-1} K^T W_{F,k} \mathbf{s} \end{aligned}$$

The matrix inversion is achieved using the Conjugate Gradient (CG) method.

3. RESULTS

We compare the performance of ℓ^1-TV deconvolution, computed via the IRN algorithm, with that of an alternative variational approach [12] (which we shall refer to as the BKS method) combining an approximate ℓ^1 data fidelity term with the Mumford-Shah regularization term [16]. The BKS results are computed using the Matlab code written by the authors of [12], and the ℓ^1-TV results are computed using the purely Matlab implementation (for a fair comparison of computation times) of the IRN algorithm from the NUMIPAD library [17]. The first test uses the 236×236 pixel Einstein image (see Fig. 1), convolved with a 3×3 pillbox kernel and subjected to salt and pepper noise. This example is identical to one of those set up by Bar *et al.* [12], allowing us to make a fair comparison by using their parameter choices for their method. The second test uses the 512×512 pixel Boat image (see Fig. 2), convolved with a 7×7 Gaussian kernel of standard deviation 2.0, and subjected to salt and pepper noise. In this case we made our own best effort to select optimal parameters for the BKS algorithm.

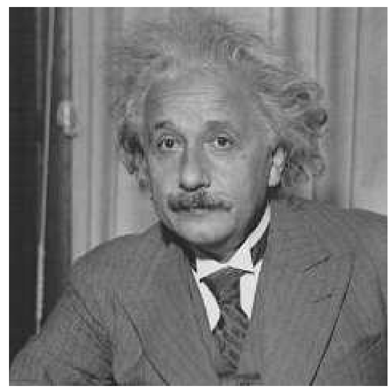


Fig. 1. Einstein test image (236×236 pixel).

Fig. 2. Boat test image (512×512 pixel).

Reconstruction SNR values and computation times are compared in Table 1, and noisy and reconstructed images are displayed in Figs. 3 to 5. The ℓ^1-TV results are significantly better for both noise levels of the Einstein image, exhibiting both less residual noise and less blur. In the 30% noise case in particular, the visual quality of the ℓ^1-TV result is remarkable, considering the level of degradation to the test image. A similar comparison holds for the Boat image (note the SNR values), but the visual difference, while obvious when viewed at full size on a display unit, is much less apparent in the small printed version in Fig. 5(b). Computation times for both methods are very similar.

		SNR (db)		Time (s)	
Image	Noise	BKS	ℓ^1-TV	BKS	ℓ^1-TV
Einstein	10%	7.9	20.5	58	55
	30%	2.2	15.8	57	50
Boat	10%	9.3	20.1	282	356
	30%	9.7	16.5	282	289

Table 1. Deconvolution performance comparison between BKS [12] method and ℓ^1-TV, computed via the IRN algorithm, on the Einstein and Boat test images.

4. CONCLUSIONS

The recent IRN algorithm is a computationally efficient approach to minimizing the ℓ^1-TV functional for general inverse problems, which are not addressed by most alternative ℓ^1-TV algorithms, which concentrate on the denoising problem. Computations times for the problem of deconvolution of images with salt and pepper noise are similar to those of the BKS algorithm, but the ℓ^1-TV functional appears to provide significantly better results than the ℓ^1-Mumford-Shah functional minimized by the BKS algorithm.

5. ACKNOWLEDGMENT

The authors thank Leah Bar for providing us with the Matlab code implementing the BKS algorithm [12].

6. REFERENCES

[1] L. Rudin, S. J. Osher, and E. Fatemi, "Nonlinear total variation based noise removal algorithms.," *Physica D*, vol. 60, no. 1-4, pp. 259–268, Nov. 1992.

[2] C. R. Vogel and M. E. Oman, "Iterative methods for total variation denoising," *SIAM J. Sci. Comput.*, vol. 17, no. 1-4, pp. 227–238, Jan. 1996.

[3] T. Chan, S. Esedoglu, F. Park, and A. Yip, "Recent developments in total variation image restoration," in *The Handbook of Mathematical Models in Computer Vision*, N. Paragios, Y. Chen, and O. Faugeras., Eds. Springer, 2005.

[4] S. Alliney, "Digital filters as absolute norm regularizers," *IEEE Trans. Signal Process.*, vol. 40, no. 6, pp. 1548–1562, 1992.

[5] M. Nikolova, "Minimizers of cost-functions involving nonsmooth data-fidelity terms. application to the processing of outliers," *SIAM J. Numerical Analysis*, vol. 40, no. 3, pp. 965–994, 2002.

[6] T. F. Chan and S. Esedoglu, "Aspects of total variation regularized L^1 function approximation," *SIAM J. Appl. Math.*, vol. 65, no. 5, pp. 1817–1837, 2005.

[7] J. F. Aujol, G. Gilboa, T. Chan, and S. Osher, "Structure-texture image decomposition - modeling, algorithms, and parameter selection," *Int. J. Comput. Vision*, vol. 67, no. 1, pp. 111–136, 2006.

[8] J. Darbon and M. Sigelle, "Image restoration with discrete constrained total variation part I: Fast and exact optimization," *J. Math. Imaging and Vision*, vol. 26, no. 3, pp. 261–276, 2006.

[9] D. Goldfarb and W. Yin, "Parametric maximum flow algorithms for fast total variation minimization," Tech. Rep. CAAM TR07-09, Department of Computational and Applied Mathematics, Rice University, 2007.

[10] B. Wohlberg and P. Rodríguez, "An iteratively reweighted norm algorithm for minimization of total variation functionals," *IEEE Signal Process Lett.*, vol. 14, no. 12, pp. 948–951, Dec. 2007.

[11] P. Rodríguez and B. Wohlberg, "Efficient minimization method for a generalized total variation functional," *IEEE Trans. Image Process.*, 2009, doi:10.1109/TIP.2008.2008420.

[12] L. Bar, N. Kiryati, and N. Sochen, "Image deblurring in the presence of impulsive noise," *Int. J. Comput. Vision*, vol. 70, no. 3, pp. 279–298, Dec. 2006.

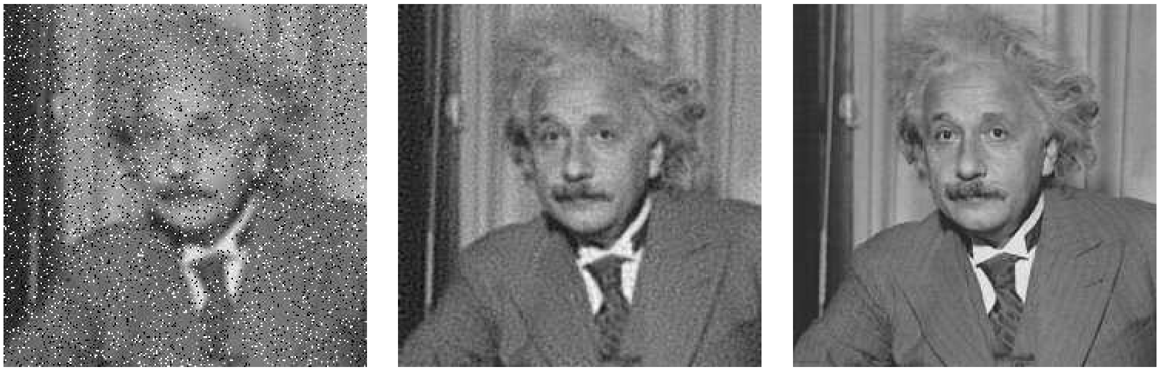
[13] A. E. Beaton and J. W. Tukey, "The fitting of power series, meaning polynomials illustrated on band-spectroscopic data," *Technometrics*, , no. 16, pp. 147–185, 1974.

[14] J. A. Scales and A. Gersztenkorn, "Robust methods in inverse theory," *Inverse Problems*, vol. 4, no. 4, pp. 1071–1091, Oct. 1988.

[15] K. P. Bube and R. T. Langan, "Hybrid ℓ^1/ℓ^2 minimization with applications to tomography," *Geophysics*, vol. 62, no. 4, pp. 1183–1195, Jul-Aug 1997.

[16] D. Mumford and J. Shah, "Optimal approximations by piecewise smooth functions and associated variational-problems.," *Comm. Pure Appl. Math.*, vol. 42, no. 5, pp. 577–685, 1989.

[17] P. Rodríguez and B. Wohlberg, "Numerical methods for inverse problems and adaptive decomposition (NUMIPAD)," Software library available from `http://numipad.sourceforge.net/`.



(a) Blur and 10% salt and pepper noise (b) BKS reconstruction (c) ℓ^1-TV IRN reconstruction

Fig. 3. Deconvolution with 10% salt and pepper noise.

(a) Blur and 30% salt and pepper noise (b) BKS reconstruction (c) ℓ^1-TV IRN reconstruction

Fig. 4. Deconvolution with 30% salt and pepper noise.

(a) Blur and 30% salt and pepper noise (b) BKS reconstruction (c) ℓ^1-TV IRN reconstruction

Fig. 5. Deconvolution with 30% salt and pepper noise.