A GENERALIZED VECTOR-VALUED TOTAL VARIATION ALGORITHM

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ABSTRACT

We propose a simple but flexible method for solving the generalized vector-valued TV (VTV) functional, which includes both the ℓ^2-VTV and ℓ^1-VTV regularizations as special cases, to address the problems of deconvolution and denoising of vector-valued (e.g. color) images with Gaussian or salt-and-pepper noise. This algorithm is the vectorial extension of the Iteratively Reweighted Norm (IRN) algorithm [1] originally developed for scalar (grayscale) images. This method offers competitive computational performance for denoising and deconvolving vector-valued images corrupted with Gaussian (ℓ^2-VTV case) and salt-and-pepper noise (ℓ^1-VTV case).

Index Terms— Vector-valued Total Variation, Color image processing

1. INTRODUCTION

While a variety of numerical algorithms for vector-valued regularization has be proposed [2, 3, 4, 5, 6, 7, 8], the method proposed in the present paper is based on the Total Variation (TV) minimization scheme for deblurring color images, first introduced in [9]. Most of the publications have focused on the Gaussian noise model, and to the best of our knowledge, [5] is the only published paper to explicity consider the salt-and-pepper noise model for *color images* within a variational framework.

The ℓ^2 vector-valued TV (VTV) regularized solution (with coupled-channel regularization [10]) of the inverse problem involving color image data $\mathbf{b}$ and forward linear operator A is the minimum of the functional

$$T(\mathbf{u}) = \frac{1}{p}\left\| A\mathbf{u} - \mathbf{b} \right\|_p^p + \frac{\lambda}{q}\left\| \sqrt{\sum_{n\in C} (D_x \mathbf{u}_n)^2 + (D_y \mathbf{u}_n)^2} \right\|_q^q, \quad (1)$$

for $p = 2$, $q = 1$, $n \in C = \{r, g, b\}$ (note that C could represent an arbitary number of channels) and notation as follows:

- $\mathbf{u}_n$ ($n \in C$) is a 1-dimensional (column) or 1D vector that represents a 2D grayscale image obtained via any ordering (although the most reasonable choices are row-major or column-major) of the image pixels.
- $\mathbf{u} = [(\mathbf{u}_r)^T (\mathbf{u}_g)^T (\mathbf{u}_b)^T]^T$ is a 1D (column) vector that represents a 2D color image.
- $\frac{1}{p}\|A\mathbf{u} - \mathbf{b}\|_p^p$ is the data fidelity term. For the scope of this paper, the linear operator A is assumed to be decoupled, i.e. A is a diagonal block matrix with elements A_n and $n \in C = \{r, g, b\}$,
- $\frac{1}{q}\left\| \sqrt{\sum_{n\in C} (D_x \mathbf{u}_n)^2 + (D_y \mathbf{u}_n)^2} \right\|_q^q$ is the generalization of TV regularization to color images with coupled channels (see [10, Section 9], also used in [5, 7, 8]),
- the p-norm of vector $\mathbf{u}$ is denoted by $\|\mathbf{u}\|_p$,
- scalar operations applied to a vector are considered to be applied element-wise, so that, for example, $\mathbf{u} = \mathbf{v}^2 \Rightarrow u[k] = (v[k])^2$ and $\mathbf{u} = \sqrt{\mathbf{v}} \Rightarrow u[k] = \sqrt{v[k]}$,
- $\sqrt{\sum_{n\in C} (D_x \mathbf{u}_n)^2 + (D_y \mathbf{u}_n)^2}$ is the discretization of $|\nabla \mathbf{u}|$ for coupled channels (see [7, eq. (3)]), and
- horizontal and vertical discrete derivative operators are denoted by D_x and D_y respectively.

Choosing $p = 1$ (ℓ^1 norm for the fidelity term), $q = 1$ in (1) leads to ℓ^1-VTV, which can be used to remove salt-and-pepper noise in color images. We also note that if set C has only one element (i.e. $\mathbf{u}$ is a grayscale image) then (1) represents the scalar TV functional.

In this paper we present an efficient algorithm to minimize the generalized vector-valued TV functional (1) for the cases of denoising ($A = I$ in (1)) and decoupled linear operator A. This algorithm, which is a computationally efficient and flexible alternative to the extension of [11] described in [5], can handle any norm with $0 < p, q \leq 2$ (including the ℓ^2-VTV and ℓ^1-VTV as special cases) by representing the ℓ^p and ℓ^q norms by the equivalent weighted ℓ^2 norms.

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2. THE VECTOR-VALUED ITERATIVELY REWEIGHTED NORM APPROACH

2.1. Previous related work

The vector-valued IRN approach is an extension of the IRN algorithm [1], and is closely related to Iteratively Reweighted Least Squares (IRLS) method for scalar [12] and vector [13] valued problems.

2.2. Derivation

In order to replace the ℓ^p norm of the fidelity term in (1) by a weighted ℓ^2 norm we define the quadratic functional

$$Q_F^{(k)}(\mathbf{u}) = \frac{1}{2}\left\| {W_F^{(k)}}^{1/2} (A\mathbf{u} - \mathbf{b}) \right\|_2^2 + \left(1 - \frac{p}{2}\right) F(\mathbf{u}^{(k)}), \tag{2}$$

where $\mathbf{u}^{(k)}$ is a constant representing the solution of the previous iteration, $F(\mathbf{u}) = \frac{1}{p}\|A\mathbf{u} - \mathbf{b}\|_p^p$, and

$$W_F^{(k)} = \operatorname{diag}\left(\tau_{F,\epsilon_F}(A\mathbf{u}^{(k)} - \mathbf{b})\right). \tag{3}$$

The function

$$\tau_{F,\epsilon_F}(x) = \begin{cases} |x|^{p-2} & \text{if } |x| > \epsilon_F \\ \epsilon_F^{p-2} & \text{if } |x| \le \epsilon_F, \end{cases}$$

is defined to avoid numerical problems when $p < 2$ and $A\mathbf{u}^{(k)} - \mathbf{b}$ has zero-valued components.

The replacement of the ℓ^q norm of the regularization term

$$R(\mathbf{u}) = \|\mathbf{r}\|_q^q, \;\; \mathbf{r} = \sqrt{\sum_{n\in\{r,g,b\}} (D_x\mathbf{u}_n)^2 + (D_y\mathbf{u}_n)^2}$$

in (1) by a weighted ℓ^2 norm is not as straightforward. We use the notation $(\mathbf{v}_n)^2 = (D_x\mathbf{u}_n)^2 + (D_y\mathbf{u}_n)^2$, define $\Phi_n = \operatorname{diag}\left(\frac{1}{\mathbf{v}_n}\right) = \operatorname{diag}\left(\frac{1}{\sqrt{(D_x\mathbf{u}_n)^2+(D_y\mathbf{u}_n)^2}}\right)$, and note that $\|\mathbf{v}_n\|_1 = \left\| \begin{bmatrix} \Phi_n^{0.5} & 0 \\ 0 & \Phi_n^{0.5} \end{bmatrix} \begin{bmatrix} D_x\mathbf{u}_n \\ D_y\mathbf{u}_n \end{bmatrix} \right\|_2^2$, neglecting possible divisions by zero. Furthermore, we may write $\|\mathbf{v}_n\|_1 = \left\| \begin{matrix} \Phi_n^{0.5}D_x\mathbf{u}_n \\ \Phi_n^{0.5}D_y\mathbf{u}_n \end{matrix} \right\|_2^2 = \|\Phi_n^{0.5}D_x\mathbf{u}_n\|_2^2 + \|\Phi_n^{0.5}D_y\mathbf{u}_n\|_2^2$. This is the key idea in [1] for solving the generalized TV for grayscale images.

For the present case of vector-valued images we first define $\Phi = \operatorname{diag}\left(\frac{1}{\sqrt{(\mathbf{v}_r)^2+(\mathbf{v}_g)^2+(\mathbf{v}_b)^2}}\right)$, then the regularization term with $q = 1$ can be expressed as

$$R(\mathbf{u}) = \|\mathbf{r}\|_1 = \left\| \left[I_3 \otimes \Phi^{0.5}\right] \left[\mathbf{v}_r^T \; \mathbf{v}_g^T \; \mathbf{v}_b^T\right]^T \right\|_2^2, \tag{4}$$

where I_N is a $N \times N$ identity matrix and $\otimes$ is the Kronecker product. Note that only Φ would change for $q \neq 1$ and therefore (4) is valid for the general case. Defining $\Phi[k,k] = \phi[k]$, it is easy to check that $\|\Phi^{0.5}\mathbf{v}_n\|_2^2 = \sum_k \phi[k]\left(\mathbf{v}_n[k]\right)^2 = \sum_k \phi[k]\left(D_x\mathbf{u}_n[k]\right)^2 + \sum_k \phi[k]\left(D_y\mathbf{u}_n[k]\right)^2$.

Using the previous equations we may write $\|\Phi^{0.5}\mathbf{v}_n\|_2^2 = \|\Phi^{0.5}D_x\mathbf{u}_n\|_2^2 + \|\Phi^{0.5}D_y\mathbf{u}_n\|_2^2$ and thus

$$R(\mathbf{u}) = \left\|W_R^{0.5}D\mathbf{u}\right\|_2^2, \tag{5}$$

where $W_R = I_6 \otimes \Phi$ and $D = I_3 \otimes [Dx^T Dy^T]^T$. It is important to emphasize that (5) is not the anisotropic separable approximation.

Now, we may replace the ℓ^q norm of the regularization term in (1) by a weighted ℓ^2 norm by means of

$$Q_R^{(k)}(\mathbf{u}) = \frac{1}{2}\left\| {W_R^{(k)}}^{1/2} D\mathbf{u} \right\|_2^2 + \left(1 - \frac{q}{2}\right) R(\mathbf{u}^{(k)}), \tag{6}$$

where $\mathbf{u}^{(k)}$ is a constant representing the solution of the previous iteration. As in the case of the data fidelity term, care needs to be taken when $q < 2$ and $\mathbf{v}_n^2 = (D_x\mathbf{u}_n)^2 + (D_y\mathbf{u}_n)^2$ has zero-valued components. We therefore define

$$\tau_{R,\epsilon_R}(x) = \begin{cases} |x|^{(q-2)/2} & \text{if } |x| > \epsilon_R \\ \epsilon_R^{(q-2)/2} & \text{if } |x| \le \epsilon_R, \end{cases} \tag{7}$$

and set $\Phi^{(k)} = \operatorname{diag}\left(\tau_{R,\epsilon_R}\left((\mathbf{v}_r^{(k)})^2 + (\mathbf{v}_b^{(k)})^2 + (\mathbf{v}_b^{(k)})^2\right)\right)$. The threshold values ϵ_F and ϵ_R may be automatically adapted to the input image; for details see Section IV.G in [1].

2.3. Algorithm

The vector-valued IRN algorithm for a general case is summarized in Algorithm 1. As for scalar IRN [1] there is a significative variant for the denoising-only case, which delivers both improved time and SNR performance than the general case, furthermore it has a better SNR / accuracy ratio (details [1, Sec. IV.E] are omitted here due to space constraints). While not done so here, this algorithm can easily be extended to handle a coupled forward operator A in (1).

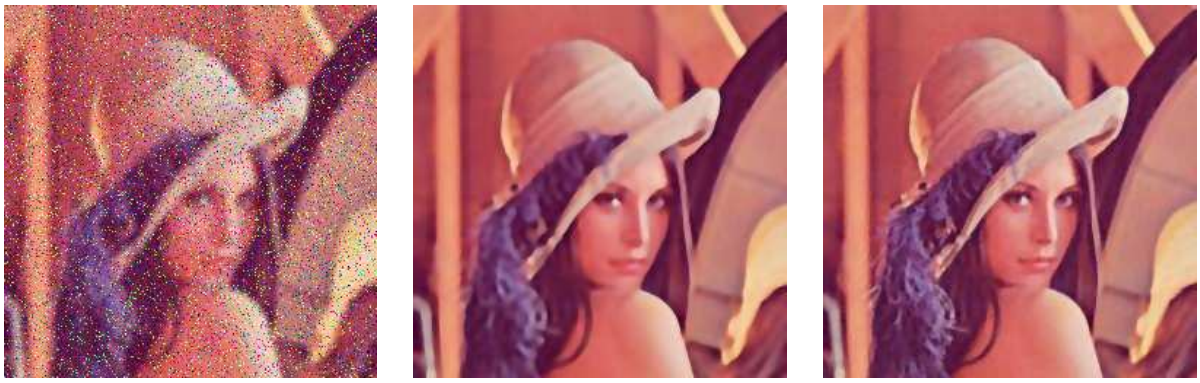
3. RESULTS

We compare the performance of the vector-valued IRN algorithm for ℓ^1 and ℓ^2 VTV denoising and deconvolution with that of three alternative variational approaches, which we refer to as ℓ^1/ℓ^2-MS[1], ℓ^1/ℓ^2-VLD[2], and FDM[3]. The test color images are the "Lena" (256×256 pixel), "Peppers" and "Mandrill" (both 512×512 pixel) images. All simulations have been carried out using Matlab-only code on a 1.83GHz Intel Dual core CPU. Results corresponding to the

[1] An approximation of the *Mumford-Shah* functional, see (9) in [5].

[2] *Vectorial lagged diffusivity*, an extension of [11] used in [5] for VTV.

[3] See [14], an implementation of the *fast dual minimization* of VTV [7].



(a) Blur and 10% salt and pepper noise (b) ℓ^1-MS [5] reconstruction (c) ℓ^1-VTV IRN reconstruction

Fig. 1. Deconvolution with 10% salt and pepper noise and with Gaussian noise ($\sigma^2 = 10^{-5}$) via [5] and VTV-IRN.

Initialize

$$\mathbf{u}^{(0)} = \left(A^T A + \lambda D^T D\right)^{-1} A^T \mathbf{b}$$

for $k = 0, 1, ...$

$$W_F^{(k)} = \text{diag}\left(\tau_{F,\epsilon_F}(A\mathbf{u}^{(k)} - \mathbf{b})\right)$$

$$\Phi^{(k)} = \text{diag}\left(\tau_{R,\epsilon_R}\left((\mathbf{v}_r^{(k)})^2 + (\mathbf{v}_b^{(k)})^2 + (\mathbf{v}_b^{(k)})^2\right)\right)$$

$$W_R^{(k)} = I_6 \otimes \Phi^{(k)}$$

$$\mathbf{u}^{(k+1)} = \left(A^T W_F^{(k)} A + \lambda D^T W_R^{(k)} D\right)^{-1} A^T W_F^{(k)} \mathbf{b}$$

end

Algorithm 1: Vector-valued IRN algorithm.

vector-valued IRN algorithm presented here may be reproduced using the the NUMIPAD (v. 0.22) distribution [15], an implementation of IRN and related algorithms.

		SNR (dB)			Time (s)		
Image	Noise	ℓ^1-MS	ℓ^1-VLD	IRN	ℓ^1-MS	ℓ^1-VLD	IRN
Lena	10%	11.1	11.2	21.8	376	92	79.2
	30%	11.1	11.0	19.2	375	91	82.9

Table 1. Deconvolution performance comparison between ℓ^1-MS and ℓ^1-VLD [5] methods and vector-valued IRN algorithm (ℓ^1-VTV case), on the "Lena" test color image.

The "Lena" image was used for the deconvolution case (to match one of the experiments described in [5]) and was blurred by a 7×7 out-of-focus kernel (generated by the Matlab command `fspecial('disk',3.2)`) and then corrupted with Gaussian additive noise or salt-and-pepper noise. Reconstruction SNR values and computation times are compared in Table 1 (deconvolution with the salt-and-pepper noise model) and Table 2 (deconvolution with the Gaussian noise model), and noisy and reconstructed images are displayed in Figs. 1(a) to 1(c) for deconvolution with the salt-and-pepper noise model. The vector-valued IRN has a better computational performance and gives significantly better results, both in terms of SNR and visual quality than the ℓ^1/ℓ^2-MS or ℓ^1/ℓ^2-VLD methods described in [5]. (Despite using the parameters values in [5, Table IV], we obtained different performance results, which we assume is due to differences in other parameters in the code provided to us.)

		SNR (dB)			Time (s)		
Image	Noise (σ^2)	ℓ^2-MS	ℓ^2-VLD	IRN	ℓ^2-MS	ℓ^2-VLD	IRN
Lena	1.0e-5	10.0	10.0	19.2	270	89	20.3
	1.0e-4	6.7	7.9	17.1	363	88	16.7
	2.5e-3	1.3	7.1	14.5	367	88	14.1

Table 2. Deconvolution performance comparison between ℓ^2-MS and ℓ^2-VLD [5] methods and vector-valued IRN algorithm (ℓ^2-VTV case), on the "Lena" test color image.

The "Peppers" and "Mandrill" images were used for the denoising only case and were corrupted with Gaussian additive noise or salt-and-pepper noise. Reconstruction SNR values and computation times are shown in Table 3 and noisy and reconstructed images are displayed in Figs. 2(a) to 2(d) for ℓ^1-VTV denoising, computed via the vector-valued IRN algorithm. Table 4 presents the results of comparing the vector-valued IRN and FDM for the ℓ^2-VTV denoising case, where the latter has better computational performance, but SNR values and visual quality are about the same for both methods.

4. CONCLUSIONS

The vector-valued IRN algorithm gives very good reconstruction quality for the ℓ^2 and ℓ^1-VTV deconvolution/denoising problems, with a superior computational performance than the only other published ℓ^1-VTV algorithm of which we are aware (the vectorial extension of [11] used in [5]). The FDM method [7, 14] for ℓ^2-VTV denoising has a better time performance, but it can not handle the ℓ^1-VTV case or a non-trivial forward operator A.

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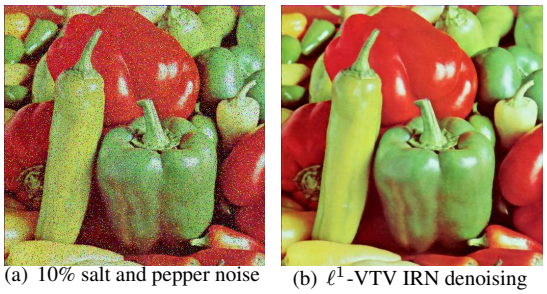
		SNR (dB)	Time (s)
Image	Noise	IRN	IRN
Peppers	10%	19.7	48.4
	30%	16.0	54.4
Mandrill	10%	10.1	46.7
	30%	8.6	53.5

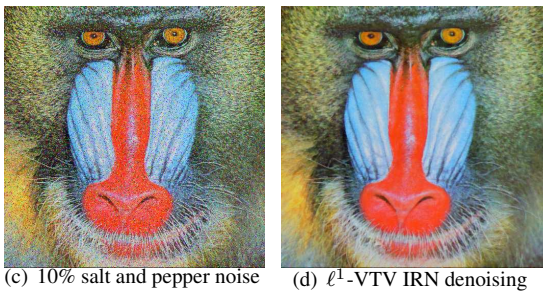
Table 3. Denoising performance results for ℓ^1-VTV, computed via the vector-valued IRN algorithm, on the "Peppers" and "Mandrill" test color images.

		SNR (dB)		Time (s)	
Image	Noise (σ^2)	FDM	IRN	FDM	IRN
Peppers	2.5e-3	20.7	20.2	7.4	12.7
	1.0e-2	17.9	17.4	13.0	15.1
Mandrill	2.5e-3	14.5	12.1	6.5	12.8
	1.0e-2	9.9	9.1	8.4	16.2

Table 4. Denoising performance results for ℓ^2-VTV, computed via the FDM [14] method and the vector-valued IRN algorithm, on the "Peppers" and "Mandrill" test color images.



(a) 10% salt and pepper noise (b) ℓ^1-VTV IRN denoising



(c) 10% salt and pepper noise (d) ℓ^1-VTV IRN denoising

Fig. 2. Denoising with 10% salt and pepper noise.

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