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The Polyphase-with-Advance Representation and Linear Phase Lifting Factorizations

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Abstract

A matrix theory is developed for the noncausal polyphase representation that underlies the theory of lifted filter banks and wavelet transforms. The theory presented here develops an extensive matrix algebra framework for analyzing and implementing linear phase two-channel filter banks via lifting cascade schemes. Whole-sample symmetric and half-sample symmetric linear phase filter banks are characterized completely in terms of the polyphase-with-advance representation, and new proofs are given of linear phase lifting factorization theorems for these two principal classes of linear phase filter banks. The theory benefits significantly from a number of group-theoretic structures arising in the polyphase-with-advance representation and in the lifting factorization of linear phase filter banks. These results form the foundations of the lifting methodology employed in Part 2 of the ISO/IEC JPEG 2000 still image coding standard.

Index Terms—Filter bank, linear phase, wavelet, polyphase, lifting, JPEG 2000.

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I. Introduction

One of the principal technologies enabling resolution-scalable transmission in the recently published ISO/IEC JPEG 2000 still image coding standard [1], [2], [3] is the discrete wavelet transform. Coding based on wavelet transforms (cascaded multirate filter banks) constitutes a significant improvement over the DCT-based coding in the original JPEG standard [4], [5]. This paper develops the "polyphase-with-advance" matrix theory that forms the signal processing foundations for the lifted wavelet transforms specified in Part 2 (Extensions) of the JPEG 2000 standard [6] (henceforth "Part 2"), on which the authors worked.

The ISO/IEC standard documents are written in "standardese" that avoids most conventional signal processing machinery, such as the convolution operation or transform analysis of linear filters. Instead, all specifications are provided in terms of explicit arithmetic formulas. This approach allows the standard to avoid normative references to external literature and minimizes the number of concepts that would otherwise have to be normatively defined within the standard, but it severely impairs the readability of the standard by digital signal processing engineers.

The present paper takes full advantage of signal processing machinery, including a new group-theoretic perspective on the polyphase representation of linear phase filter banks, to prove a fundamental result (Theorem 9): that *any* whole-sample symmetric two-channel filter bank can be factored completely using half-sample symmetric lifting filters, the structure implicit in the specifications of JPEG 2000 Part 2 Annex G. We derive the limits to which this methodology can be generalized for half-sample symmetric filter banks (Theorem 14), a result that led directly to the considerably more general lifting machinery specified in Part 2 Annex H. The polyphase-with-advance representation also forms the foundation for several other papers currently being written on the filter bank specifications in JPEG 2000 Part 2. Part of this work [7] is related to symmetric boundary extension techniques developed in [8], [9], [10], [11]. Preliminary announcements of these results were given in [12], [13].

The outline of the paper is as follows. The remainder of the Introduction reviews prior work and the definition of perfect reconstruction filter banks. Section II derives the basic matrix-vector algebra of the polyphase-with-advance representation. Section III characterizes linear phase filter banks in terms of the algebraic properties of their polyphase-with-advance representation. These characterizations are extended to lifting factorizations in Section IV, and the existence and specific form of the linear phase factors for whole-sample symmetric and half-sample symmetric filter banks are derived in Sections V and VI.

A. Comparison to Prior Work

The theory of lifted filter banks originated with Bruekers and van den Enden [14] and was subsequently rediscovered in the context of wavelet transforms and extensively developed by Sweldens and collaborators [15], [16], [17], [18]. While lifting structures have been used by some authors (e.g., [19]) as parameter spaces for numerical filter bank design, the present paper deals with the problem of factoring a given linear phase filter bank into linear phase lifting steps of prescribed form, following the definition of lifting presented by Daubechies and Sweldens [17]. This covers the situation created by the JPEG 2000 standard, which allows users to employ filter banks designed by any method (e.g., the structures introduced in [20]), provided their implementation is specified in terms of a lifting factorization. The existence of factorizations with linear phase lifting steps is also exploited by efficient hardware implementations of lifted wavelet transforms [21].

Recent work on filter bank design has focused on M-channel systems, including linear phase systems [22], [23], [24] and lifting structures for M-channel filter banks and regular M-band wavelets [25], [26]. In particular, several papers have combined lifting with other lattice structures that structurally enforce linear phase [27], [28], [29]. In [23], linear phase matrices were decomposed into first-order polyphase factors, $\mathbf{G}_i(z)$, having the form

$$\mathbf{G}_i(z) = \mathbf{\Phi}_i \, \mathbf{W} \, \mathbf{\Lambda}(z) \, \mathbf{W} \, ,$$

where $\mathbf{\Lambda}(z)$ is a first-order diagonal delay matrix, $\mathbf{W}$ is an orthogonal matrix containing only ± 1's, and $\mathbf{\Phi}_i$ is a block-diagonal constant matrix containing all of the degrees of freedom.

Following [14], Tran [27] factored each nonzero block, $\mathbf{V}$, in $\mathbf{\Phi}_i$ into a lower-upper decomposition, $\mathbf{V} = \mathbf{PDLU}$, and then decomposed $\mathbf{L}$, $\mathbf{U}$, and the diagonal matrix, $\mathbf{D}$, into elementary matrices, which correspond to constant (zeroth-order) lifting filters. ($\mathbf{P}$ is a permutation matrix.) This approach was subsequently extended to oversampled filter banks [28] and filter banks satisfying vanishing-moment conditions [29]. In all of these papers, the delays in the system are encapsulated in diagonal delay matrices, $\mathbf{\Lambda}(z)$, and thereby isolated from the (constant) lifting operations. This approach to lifting is very different from that defined in [17], and it sidesteps all questions about the symmetries of higher-order lifting filters that arise in consideration of the lifting framework specified in [6]. Consequently, these recent works on very general classes of M-channel filter banks have not answered certain basic questions about the factorization of two-channel filter banks into linear phase lifting steps, such as those raised by Part 2 of the JPEG 2000 standard.

A tremendous amount of "translation" of JPEG 2000 standardese into conventional engineering language has been performed by Taubman and Marcellin in their recent textbook [3], but that volume principally addresses Part 1 (the "Baseline") of the JPEG 2000 standard. Since Part 2 extensions are only mentioned briefly in [3], our goal is to provide a thorough treatment of the linear phase filter bank theory behind the Part 2 wavelet transform algorithms. We develop the polyphase-with-advance representation significantly beyond the treatment given in [17] or [3] and characterize lifting factorizations for two-channel linear phase filter banks. This includes a new group theory for lifted whole-sample symmetric filter banks, characterizations of linear phase lifting steps and their effects on linear phase input signals, and rigorous existence proofs for lifting factorizations using linear phase lifting filters. We provide a thorough treatment of half-sample symmetric filter banks, which are not covered in [3]. Another benefit of the polyphase-with-advance theory is that it enables us to dispense with the time-varying convolution operation used to establish a number of results in [3] and prove or generalize them using ordinary convolution and conventional polyphase structures.

B. Perfect Reconstruction Multirate Filter Banks

This paper studies two-channel multirate digital filter banks [30], [31], [32], [33], [34], [35], [36] of the form shown in Figure 1. We only consider systems in which both the analysis filters, $H_0(z)$, $H_1(z)$, and the synthesis filters, $G_0(z)$, $G_1(z)$, are FIR linear time-invariant filters with real impulse responses. Such a system is called a *perfect reconstruction multirate filter bank* (abbreviated to just "filter bank" in this paper) if it has a transfer function satisfying

$$\frac{\hat{X}(z)}{X(z)} = Az^{-D} \tag{1}$$

for some integer D and some constant $A \neq 0$. When a filter bank corresponds to (continuous-time) wavelets [31], [37], [38], [39], [40], [41], the moment conditions necessary for wavelet regularity imply that one of the filters must be lowpass, while the other must be highpass. Since image coding applications usually use wavelet filter banks, we shall speak of H_0 and G_0 as lowpass filters and regard H_1 and G_1 as highpass filters, although none of the arguments in this paper depends on bandpass characteristics. The paper [17] established a convention of using noncausal polyphase matrices whose determinant is delay-free to define lifting factorizations, which results in nonrealizable representations of filter banks. While this is undesirable from an implementation perspective, we shall follow this convention, which was adopted in [1], [6], as we derive the fundamental lifting algorithms.

II. The Noncausal Polyphase-with-Advance Representation

Noncausal lifting is based on the polyphase representation shown in Figure 2. Note the use of an advance when splitting the input signal into its polyphase components in the analysis bank. This particular analysis-synthesis structure was described previously in [33] and labelled "polyphase with advance" in [34]. It is much less common in the filter bank literature, however, than realizable polyphase structures in which demultiplexing of the input signal is represented causally. To facilitate working with noncausal filter bank structures, we therefore begin by developing the matrix theory for "polyphase with advance."

A. Polyphase-with-Advance Matrix-Vector Arithmetic

The polyphase components, x_0 and x_1, of a signal are defined to be

$$x_i(n) = x(2n+i) \quad , \quad i = 0, 1. \tag{2}$$

This corresponds to the transform-domain decomposition in Figure 2,

$$X(z) = X_0(z^2) + z^{-1}X_1(z^2) \; . \tag{3}$$

The *polyphase vector form* of a signal is defined to be

$$\mathbf{x}(n) = \begin{bmatrix} x_0(n) \\ x_1(n) \end{bmatrix} \quad ; \quad \mathbf{X}(z) = \begin{bmatrix} X_0(z) \\ X_1(z) \end{bmatrix} \; . \tag{4}$$

Observe that the synthesis bank in Figure 2 uses multiplex-with-delay. Since this is the mathematical inverse of the demultiplex-with-advance operation used in the analysis bank, "analysis" and "synthesis" polyphase components and vectors are related to the scalar input and output signals by the same formulas; i.e., (2) also applies to $\hat{x}(n)$:

$$\hat{x}_i(n) \;=\; \hat{x}(2n+i) \quad , \quad i = 0, 1 \, , \tag{5}$$

$$\hat{X}(z) \;=\; \hat{X}_0(z^2) + z^{-1}\hat{X}_1(z^2) \; . \tag{6}$$

This is not true of other polyphase representations, such as the representation defined in [32] that employs both demultiplex-with-delay and multiplex-with-delay.

$\mathbf{H}_a(z)$ and $\mathbf{G}_s(z)$ are the analysis and synthesis polyphase matrices, respectively, in Figure 2:

$$\mathbf{H}_a(z) \equiv \begin{bmatrix} H_{a_{00}}(z) & H_{a_{01}}(z) \\ H_{a_{10}}(z) & H_{a_{11}}(z) \end{bmatrix}, \tag{7}$$

$$\mathbf{G}_s(z) \equiv \begin{bmatrix} G_{s_{00}}(z) & G_{s_{01}}(z) \\ G_{s_{10}}(z) & G_{s_{11}}(z) \end{bmatrix}. \tag{8}$$

We derive the necessary relationship between $\mathbf{H}_a(z)$ and the analysis filters in Figure 1 by equating the analysis banks in Figures 1 and 2, as shown in Figure 3:

$$\begin{bmatrix} H_0(z)X(z) \\ H_1(z)X(z) \end{bmatrix} = \begin{bmatrix} H_{a_{00}}(z^2)X(z) + zH_{a_{01}}(z^2)X(z) \\ H_{a_{10}}(z^2)X(z) + zH_{a_{11}}(z^2)X(z) \end{bmatrix}.$$

This yields the following relationships:

$$H_i(z) = H_{a_{i0}}(z^2) + zH_{a_{i1}}(z^2) \quad , \quad i = 0, 1, \tag{9}$$

$$h_{a_{ij}}(n) = h_i(2n - j) \quad , \quad i, j = 0, 1. \tag{10}$$

Similarly, equating the synthesis formulas corresponding to Figures 1 and 2 leads to the following relationships between the synthesis filters, $G_j(z)$, and the entries in $\mathbf{G}_s(z)$:

$$G_j(z) = G_{s_{0j}}(z^2) + z^{-1}G_{s_{1j}}(z^2) \quad , \quad j = 0, 1, \tag{11}$$

$$g_{s_{ij}}(n) = g_j(2n + i) \quad , \quad i, j = 0, 1. \tag{12}$$

This approach to defining analysis and synthesis transfer matrices for polyphase-with-advance seems conceptually simpler than the approach in [17], in which the polyphase analysis matrix is defined as the time-reversed transpose of the synthesis polyphase matrix decomposition (11) of time-reversed analysis filters, $\tilde{h}(z^{-1})$ and $\tilde{g}(z^{-1})$. The reader can verify that (9) generates the same analysis matrix as the definition in [17]. Note that neither the analysis polyphase matrix given by (9) nor the synthesis matrix (11) are the same as the "type-1/type-2 polyphase representations" defined in [32, Sections 4.3 and 5.5]. This is a consequence of the anticausal demultiplexing (i.e., the advance) in the polyphase-with-advance analysis bank and the application of a delay to the *highpass* channel in the synthesis bank.

B. Perfect Reconstruction

The standard polyphase condition for characterizing perfect reconstruction (1) for FIR filter banks, which includes the case of polyphase-with-advance, is

$$\det \mathbf{H}_a(z) = az^{-d} \quad , \quad a \neq 0\ . \tag{13}$$

This holds if and only if $\mathbf{H}_a(z)$ is a FIR filter bank with FIR inverse, $\mathbf{H}_a^{-1}(z)$. In [17], however, the authors state there is no loss of generality in assuming that $a = 1$ and $d = 0$:

$$\det \mathbf{H}_a(z) = 1\ . \tag{14}$$

Unfortunately, this delay-free normalization leads to noncausal filters. For instance, it is known [32, Theorem 14.7.1] that the McMillan degree of a causal paraunitary filter bank is equal to its determinantal degree (d in (13)). In Section III we show that similar noncausality results follow for linear phase filter banks as well. In fact, the only causal FIR filter banks satisfying (14) are *unimodular* filter banks [32]—causal FIR filter banks with causal FIR inverses. The class of unimodular filter banks, however, does not include most of the filter banks of interest to source coding applications, such as paraunitary or linear phase filter banks.

In order to accommodate noncausality, we write (noncausal) FIR filter banks in the transform domain in terms of Laurent polynomials; i.e., polynomials in both z and z^{-1}:

$$F(z) = \sum_{n=a}^{b} f(n)\, z^{-n}\ ,$$

where it may be that $a < 0$. The *support interval* of an FIR filter, denoted

$$\mathrm{supp}(f) = [a, b]\ ,$$

is defined to be the smallest closed interval containing all of the filter's nonzero impulse response coefficients. Equivalently, it is the largest closed interval for which $f(a) \neq 0$ and $f(b) \neq 0$. The *order* of a filter supported in the interval $[a, b]$ is defined to be

$$\mathrm{order}(F) \equiv b - a\ . \tag{15}$$

The characterization (13) of perfect reconstruction has an important consequence: FIR filter banks form a group. Since the determinant is multiplicative, we immediately get

Theorem 1 (FIR Filter Bank Group Property): The set of Laurent polynomial matrices satisfying (13) forms a group (the *FIR filter bank group*) under matrix multiplication. The matrices satisfying the Daubechies-Sweldens normalization (14) form a normal subgroup of the FIR filter bank group.

C. Structures for Interleaved Filtering

One convenient feature of critically sampled causal filter banks is that they can be performed *in situ*; i.e., the output can overwrite the input as it is consumed. If the transform is noncausal but FIR, as in this paper, then things get a bit more complicated, but conceptually one can regard the transform as writing an output signal of length N onto the memory occupied by an input signal of length N. This is the perspective taken in JPEG 2000, where lowpass subband coefficients are regarded as “written over” the even-indexed input samples and highpass coefficients are interleaved between them on the odd-indexed samples.

We can describe interleaved analysis bank output via the block diagram in Figure 4. The scalar output signal, y, consists of interleaved lowpass and highpass samples:

$$y(2k) = y_0(k) \quad , \quad y(2k+1) = y_1(k) \; .$$

At each time k (in the polyphase domain), the lowpass sample, $y_0(k)$, is output *before* the corresponding highpass sample, $y_1(k)$. For this reason, the interleaved filtering convention in the JPEG 2000 standard is sometimes described as a “lowpass-first” convention. Note that the use of the polyphase representation for the analysis bank in Figure 4 is not essential since the subbands Y_0 and Y_1 are mathematically identical to those produced in Figure 1; i.e., the analysis bank in Figure 4 also could have been drawn as in Figure 1.

III. Polyphase Characterizations of Linear Phase

It was proven in [20] that the only nontrivial classes of two-channel linear phase FIR filter banks are the *whole-sample symmetric* (WS) and *half-sample symmetric* (HS) classes shown in Figure 5. Both filters in a WS filter bank have odd lengths and are symmetric. In an HS filter bank both filters have even lengths; the lowpass filter is symmetric about the half-integer midway between its two middle samples while the highpass filter is antisymmetric.

A. Linear Phase Signals

It is easy to show that a discrete-time signal, $x(n)$, is symmetric about $i_0 \in \mathbb{Z}/2$ (a whole or half integer) if and only if

$$X(z^{-1}) = z^{2i_0} X(z) \; . \tag{16}$$

Similarly, $x(n)$ is antisymmetric about $i_0 \in \mathbb{Z}/2$ if and only if

$$X(z^{-1}) = -z^{2i_0} X(z) \; . \tag{17}$$

What about symmetry for the polyphase components of a symmetric signal, x? First, consider the case where $i_0 \in \mathbb{Z}$ (a whole integer) and apply (16) to (3):

$$X_0(z^{-2}) + zX_1(z^{-2}) = z^{2i_0}(X_0(z^2) + z^{-1}X_1(z^2)) \; . \tag{18}$$

Since $2i_0$ is even, equating the even (resp., odd) terms on both sides of (18) shows that $X_0(z^2)$ (resp., $z^{-1}X_1(z^2)$) must be symmetric about i_0. This means that $X_0(z)$ is symmetric about $i_0/2$ and $X_1(z)$ is symmetric about $(i_0 - 1)/2$:

$$X_0(z^{-1}) = z^{i_0}X_0(z) \quad ; \quad X_1(z^{-1}) = z^{i_0-1}X_1(z) \; . \tag{19}$$

Conversely, (19) implies (16). Define the diagonal delay matrix,

$$\mathbf{\Lambda}(z) \equiv \mathrm{diag}(1, z^{-1}) \; .$$

Then whole-sample symmetry of x about $i_0 \in \mathbb{Z}$ can be expressed equivalently in terms of the transform-domain analysis polyphase vector as

$$\mathbf{X}(z^{-1}) = z^{i_0}\mathbf{\Lambda}(z)\mathbf{X}(z) \; . \tag{20}$$

By (6) it follows that (20) also characterizes whole-sample symmetry of reconstructed signals in terms of the symmetries of their synthesis polyphase components.

Now suppose that i_0 is an odd multiple of $1/2$. Since $2i_0$ is odd, (19) is replaced by

$$X_0(z^{-1}) = z^{(2i_0-1)/2}X_1(z) \quad ; \quad X_1(z^{-1}) = z^{(2i_0-1)/2}X_0(z) \; , \tag{21}$$

which are equivalent to one another. In this case, x_0 and x_1 are *not* symmetric; instead, (21) says that x_0 and x_1 are “mirror images,” or time-reversed versions, of one another. Define

$$\mathbf{J} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; ; \tag{22}$$

then half-sample symmetry of x about i_0 can be expressed equivalently as

$$\mathbf{X}(z^{-1}) = z^{(2i_0-1)/2}\mathbf{J}\mathbf{X}(z) \; . \tag{23}$$

By (6) it follows that (23) also characterizes half-sample symmetry of reconstructed signals in terms of the mirror-image property for their synthesis polyphase components.

B. Whole-Sample Symmetric Filter Banks

Suppose $H(z)$ is a WS filter with analysis polyphase components $H_{a_0}(z)$ and $H_{a_1}(z)$, as defined by (9). Let the integer γ denote the midpoint (the axis of symmetry) for the impulse response, and note that γ is equal to the filter's group delay. Just as in the analysis leading to (19), the even polyphase component, $H_{a_0}(z)$, is symmetric about $\gamma/2$. In light of (9), however, the odd polyphase component, $H_{a_1}(z)$, is symmetric about $(\gamma+1)/2$. These symmetries can be written in row vector form as

$$\begin{bmatrix} H_{a_0}(z^{-1}) & H_{a_1}(z^{-1}) \end{bmatrix} = z^{\gamma} \begin{bmatrix} H_{a_0}(z) & zH_{a_1}(z) \end{bmatrix} , \tag{24}$$

which is equivalent to applying (20) to $H(z)$, with $\mathbf{\Lambda}(z)$ replaced by $\mathbf{\Lambda}(z^{-1})$ to account for the difference between (9) and (3).

Now consider a WS filter bank, $\{H_0(z), H_1(z)\}$, with group delays γ_0 and γ_1 respectively and analysis polyphase matrix (7). Applying formula (24) to the rows of the analysis polyphase matrix yields a polyphase characterization of whole-sample symmetry:

Lemma 2: $\mathbf{H}_a(z)$ is a WS analysis filter bank with group delays γ_0 and γ_1 if and only if

$$\mathbf{H}_a(z^{-1}) = \mathrm{diag}(z^{\gamma_0}, z^{\gamma_1})\mathbf{H}_a(z)\mathbf{\Lambda}(z^{-1}) . \tag{25}$$

Similar consideration of (11) shows that a synthesis bank, $\{G_0(z), G_1(z)\}$, is WS about γ_0' and γ_1' if and only if

$$\mathbf{G}_s(z^{-1}) = \mathbf{\Lambda}(z)\mathbf{G}_s(z)\mathrm{diag}(z^{\gamma_0'}, z^{\gamma_1'}) . \tag{26}$$

1) Delay-minimized WS filter banks: Assume that $\mathbf{H}_a(z)$ satisfies (13), take the determinant of both sides of (25), and equate exponents:

$$2d = \gamma_0 + \gamma_1 + 1 , \tag{27}$$

where d is the determinantal delay in (13). This prevents WS filter banks from being *concentric* [10, Section 4.1]; i.e., we cannot have $\gamma_0 = \gamma_1$. To normalize WS group delays so that $d = 0$, one can set $\gamma_0 = 0$ and $\gamma_1 = -1$. This was described as a "noncausal minimal phase" convention in [10] in an attempt to capture the idea that the group delays of the filters have been moved as close to zero as possible, subject to the constraint (27). Unfortunately, this terminology is easily confused with the widely-used notion of minimum phase filters [42].

Taubman and Marcellin [3, Section 6.1.3] define a *delay-normalized FIR filter bank* as one whose polyphase analysis matrix has a determinantal delay of zero ($d = 0$ in (13)) and whose lowpass filter is centered at $\gamma_0 = 0$ for odd-length linear phase filters or $\gamma_0 = -1/2$ for even-length linear phase filters. This captures the essential feature we're after, so we will follow Taubman and Marcellin's definition but use the term *delay-minimized filter bank* as a more-descriptive synonym for *delay-normalized filter bank*. Note that condition (27) forces the choice $\gamma_1 = -1$ for the highpass group delay in a delay-minimized WS analysis filter bank.

With $\gamma_0 = 0$ and $\gamma_1 = -1$, formula (25) becomes

$$\mathbf{H}_a(z^{-1}) = \mathbf{\Lambda}(z)\,\mathbf{H}_a(z)\,\mathbf{\Lambda}(z^{-1})\ . \tag{28}$$

This is equivalent to the definition of delay-minimized WS filter banks and implies

$$\begin{aligned}\mathbf{H}_a^{-1}(z^{-1}) &= \left\{\mathbf{\Lambda}(z)\,\mathbf{H}_a(z)\,\mathbf{\Lambda}(z^{-1})\right\}^{-1} \\ &= \mathbf{\Lambda}(z)\,\mathbf{H}_a^{-1}(z)\,\mathbf{\Lambda}(z^{-1})\ . \end{aligned} \tag{29}$$

If $\mathbf{H}_a^{-1}(z)$ is regarded as a synthesis filter bank, comparison with (26) shows that $\gamma_0' = 0$ and $\gamma_1' = 1$. An important consequence is

Theorem 3 (WS Group Property): Filter banks satisfying (28) form a subgroup (the *WS subgroup*) in the FIR filter bank group.

Proof: It is easy to verify that (28) is preserved under matrix multiplication, and (29) shows that property (28) is preserved under matrix inversion. ■

The group properties of WS filter banks in causal realizations are studied in [43].

Example 1: The identity matrix, **I**, is the polyphase matrix of a WS filter bank known as the *lazy wavelet* filter bank. The transfer functions for the lazy wavelet analysis bank are

$$H_0(z) = 1 \quad , \quad H_1(z) = z\ . \tag{30}$$

2) Interleaved output from WS filter banks: Note that the lowpass impulse response in a delay-minimized WS filter bank leads the highpass impulse response by one sample, as shown in Figure 5. If one applies a WS analysis filter bank satisfying (28) to an input signal, $x(n)$, the first lowpass output, $y_0(0)$, is a weighted average of input samples centered at $n = 0$, while the first highpass output, $y_1(0)$, is a weighted average of input samples centered at $n = 1$. This was the original motivation for associating lowpass outputs with even-indexed input samples and for thinking of this subband transform as a "lowpass first" decomposition.

The study of filter banks in [3] uses *translated impulse responses,* which, in the case of WS filter banks, are shifted so that *both* impulse responses are concentric about $n = 0$. Thus, delay-normalized WS filter banks with translated impulse responses do not satisfy (27).

This means that delay-normalized WS filter banks with translated impulse responses, as defined in [3], do *not* provide perfect reconstruction in filter bank structures that use ordinary convolution, e.g., as shown in Figure 1. To achieve perfect reconstruction, delay-normalized filter banks with translated impulse responses are implemented in [3] via a process of *time-varying convolution,* which is defined so as to generate alternating lowpass and highpass interleaved outputs. Since this results in exactly the same output as Figure 4, which uses conventionally defined convolution, Occam's razor leads us to prefer studying filter banks in terms of ordinary convolution and conventional polyphase structures. The following useful theorem about applying WS filter banks to WS input signals is proven in [3, Section 6.5.2] using time-varying convolution.

Theorem 4: Let $x(n)$ be whole-sample symmetric about $i_0 \in \mathbb{Z}$, and let $y(n)$ be the interleaved output from the analysis filter bank, $\mathbf{H}_a(z)$, in Figure 4. If $\mathbf{H}_a(z)$ is a delay-minimized WS filter bank then the interleaved output is also whole-sample symmetric about i_0.

Proof: Recall that a signal is WS about i_0 if and only if its polyphase vector satisfies (20). If $\mathbf{Y}(z)$ is the output polyphase vector, then

$$\begin{aligned}
\mathbf{Y}(z^{-1}) &= \mathbf{H}_a(z^{-1})\,\mathbf{X}(z^{-1}) \\
&= \mathbf{\Lambda}(z)\,\mathbf{H}_a(z)\,\mathbf{\Lambda}(z^{-1})\,\mathbf{X}(z^{-1}) \quad \text{by (28)} \\
&= \mathbf{\Lambda}(z)\,\mathbf{H}_a(z)\,\mathbf{\Lambda}(z^{-1})\,z^{i_0}\mathbf{\Lambda}(z)\,\mathbf{X}(z) \quad \text{by (20)} \\
&= z^{i_0}\mathbf{\Lambda}(z)\,\mathbf{Y}(z)\ .
\end{aligned}$$

By (20), $y(n)$ is whole-sample symmetric about i_0. ■

The reader can verify that Theorem 4 provides a much simpler derivation of the same symmetries that were derived by time-domain arguments in [9], [10] for WS filter banks.

C. Half-Sample Symmetric Filter Banks

Now suppose that $H(z)$ is HS about a half-integer, γ. Applying (16) to (9) and arguing as before, the mirror property for the polyphase components takes the form

$$H_{a_0}(z^{-1}) = z^{(2\gamma+1)/2} H_{a_1}(z)\ . \tag{31}$$

Using (23) with the appropriate delay factor for analysis filters, this can be written

$$\begin{bmatrix} H_{a_0}(z^{-1}) & H_{a_1}(z^{-1}) \end{bmatrix} = z^{(2\gamma+1)/2} \begin{bmatrix} H_{a_1}(z) & H_{a_0}(z) \end{bmatrix}\ . \tag{32}$$

Similarly, if $H(z)$ is half-sample *anti*-symmetric (HA) about γ then

$$H_{a_0}(z^{-1}) = -z^{(2\gamma+1)/2} H_{a_1}(z)\ . \tag{33}$$

Combine (31) and (33) to characterize HS filter banks:

Lemma 5: $\mathbf{H}_a(z)$ is an HS analysis filter bank with group delays γ_0 and γ_1 if and only if

$$\mathbf{H}_a(z^{-1}) = \operatorname{diag}\left(z^{(2\gamma_0+1)/2},\ -z^{(2\gamma_1+1)/2}\right) \mathbf{H}_a(z)\,\mathbf{J}\ . \tag{34}$$

Consideration of (11) shows that a synthesis bank is HS about γ_0' and γ_1' if and only if

$$\mathbf{G}_s(z^{-1}) = \mathbf{J}\,\mathbf{G}_s(z)\operatorname{diag}\left(z^{(2\gamma_0'-1)/2},\ -z^{(2\gamma_1'-1)/2}\right)\ . \tag{35}$$

1) Delay-minimized HS filter banks: Take the determinant of (34) and equate exponents:

$$2d = \gamma_0 + \gamma_1 + 1\ , \tag{36}$$

where d is the determinantal delay in (13). As in the WS case, $\gamma_0 + \gamma_1$ is again constrained to be an odd integer, but in contrast to the WS case this is not incompatible with the concentric condition $\gamma_0 = \gamma_1$ since γ_0 and γ_1 are odd multiples of $1/2$. It follows from the results in [10, Section 3.1.1] that *every* HS filter bank can be reduced to a concentric filter bank by applying admissible delays to the filters. For instance, the concentric delay-minimized convention from [8],

$$\gamma_0 = -1/2 = \gamma_1\ ,$$

implies $d = 0$. With this choice of group delays, equation (34) reduces to

$$\mathbf{H}_a(z^{-1}) = \mathbf{L}\,\mathbf{H}_a(z)\,\mathbf{J} \quad \text{where} \quad \mathbf{L} \equiv \operatorname{diag}(1, -1)\ . \tag{37}$$

We refer to a polyphase matrix satisfying (37) as a *concentric delay-minimized* HS analysis filter bank. The inverse filter bank satisfies

$$\mathbf{H}_a^{-1}(z^{-1}) = \mathbf{J}\,\mathbf{H}_a^{-1}(z)\,\mathbf{L}\ , \tag{38}$$

and, if $\mathbf{H}_a^{-1}(z)$ is regarded as a synthesis filter bank, comparison with (35) shows that

$$\gamma_0' = 1/2 = \gamma_1'\ .$$

In comparison, delay-normalized HS filter banks with translated impulse responses, as defined in [3], have group delays $\gamma_0 = -1/2$ and $\gamma_1 = 1/2$. This makes them incompatible with (36), implying that they do not form a perfect reconstruction filter bank using the structure in Figure 1.

In contrast to (28) and (29), the symmetry equations (37) and (38) for forward and inverse delay-minimized HS filter banks are different. Moreover, in contrast to Theorem 3, delay-minimized HS filter banks do *not* form a group. (E.g., the identity matrix is not an HS filter bank.) In fact, we can prove something stronger:

Theorem 6: Let $\mathbf{E}(z)$ be a concentric delay-minimized HS analysis filter bank of determinant 1. **(i)** $\mathbf{E}^{-1}(z)$ *never* satisfies (37). **(ii)** If $\mathbf{F}(z)$ also satisfies (37), then $\mathbf{E}(z)\,\mathbf{F}(z)$ never satisfies (37), either.

Proof: **(i)** Note that this proposition says $\mathbf{E}^{-1}(z)$ never satisfies the condition for concentric delay-minimized HS *analysis* filter banks, which prevents (37) from defining a group. If $\mathbf{E}^{-1}(z)$ *does* satisfy (37), then $\mathbf{E}(z)$ satisfies both (37) *and* (38), so

$$\mathbf{E}(z) = \mathbf{L}\,\mathbf{E}(z^{-1})\,\mathbf{J} = \mathbf{L}\,\mathbf{J}\,\mathbf{E}(z)\,\mathbf{L}\,\mathbf{J}\ .$$

This says that $E_{00}(z) = -E_{11}(z)$ and $E_{01}(z) = E_{10}(z)$, so

$$-E_{00}^2(z) - E_{01}^2(z) = \det \mathbf{E}(z) = 1\ .$$

Since the impulse response coefficients in $\mathbf{E}(z)$ are real, this equation cannot be satisfied when z is on the real axis, hence $\mathbf{E}^{-1}(z)$ cannot satisfy (37).

(ii) Suppose $\mathbf{H}(z) \equiv \mathbf{E}(z)\,\mathbf{F}(z)$ satisfies (37); then

$$\begin{aligned}
\mathbf{L}\,\mathbf{E}(z)\,\mathbf{F}(z)\,\mathbf{J} &= \mathbf{L}\,\mathbf{H}(z)\,\mathbf{J} \\
&= \mathbf{H}(z^{-1}) \\
&= \mathbf{E}(z^{-1})\,\mathbf{F}(z^{-1}) \\
&= \mathbf{L}\,\mathbf{E}(z)\,\mathbf{J}\,\mathbf{L}\,\mathbf{F}(z)\,\mathbf{J}\ .
\end{aligned}$$

Simplification implies $\mathbf{J}\,\mathbf{L} = \mathbf{I}$, a contradiction. ■

Example 2: The simplest example of an HS filter bank is the well-known Haar filter bank:

$$H_0(z) = (z+1)/2 \quad , \quad H_1(z) = z - 1\ . \tag{39}$$

2) Output subband symmetries from HS filter banks: The analogue of Theorem 4 for HS filter banks gives the subband symmetries resulting from the action of an HS filter bank on an HS input signal whose polyphase components satisfy the mirror-image property (23).

Theorem 7: Let $i_0 \in \mathbb{Z}/2$ be an odd multiple of $1/2$ and let $x(n)$ be HS about i_0. If $\mathbf{H}_a(z)$ is a concentric delay-minimized HS filter bank then the lowpass output subband, $y_0(n)$, is symmetric about $(2i_0 - 1)/4$ while the highpass output subband, $y_1(n)$, is antisymmetric about $(2i_0 - 1)/4$.

Proof: If $\mathbf{Y}(z)$ is the output polyphase vector, then

$$
\begin{aligned}
\mathbf{Y}(z^{-1}) &= \mathbf{H}_a(z^{-1})\,\mathbf{X}(z^{-1}) \\
&= \mathbf{L}\mathbf{H}_a(z)\,\mathbf{J}\mathbf{X}(z^{-1}) \quad \text{by (37)} \\
&= \mathbf{L}\mathbf{H}_a(z)\,\mathbf{J}\,z^{(2i_0-1)/2}\,\mathbf{J}\mathbf{X}(z) \quad \text{by (23)} \\
&= z^{(2i_0-1)/2}\,\mathbf{L}\mathbf{Y}(z)\,. \qquad (40)
\end{aligned}
$$

Applying (16) and (17) to the lowpass and highpass components of $\mathbf{Y}(z)$, we see that $y_0(n)$ is symmetric about $(2i_0 - 1)/4$ and that $y_1(n)$ is antisymmetric about $(2i_0 - 1)/4$. ■

The reader can verify that Theorem 7 implies the same symmetries for HS filter bank subbands that were derived in [9], [10]. In Section VI we will prove that linear phase lifting steps for HS filter banks preserve these subband symmetries, a result more in the spirit of Theorem 4. These results are used in [7] to derive lifting-domain boundary extension algorithms that are mathematically equivalent to symmetric pre-extension transforms.

IV. Lifting Decompositions

Lifting [15], [16], [17] is a lattice structure for polyphase filter banks in which the steps in the decomposition alternately "lift," or update, one channel at a time.

A. Lifting Matrices

Figure 6 depicts an analysis bank in which the first lifting step updates x_0 with a filtered version of x_1 and the second step updates x_1 with a filtered version of x_0. The lowpass update is described by an upper-triangular polyphase matrix,

$$
\mathbf{S}_0(z) = \begin{bmatrix} 1 & S_0(z) \\ 0 & 1 \end{bmatrix}, \qquad (41)
$$

while the highpass update is described by a lower-triangular matrix,

$$
\mathbf{S}_1(z) = \begin{bmatrix} 1 & 0 \\ S_1(z) & 1 \end{bmatrix}. \qquad (42)
$$

We say an upper-triangular lifting matrix has a *lowpass update characteristic* while a lower-triangular lifting matrix has a *highpass update characteristic.* The lattice in Figure 6 corresponds to the matrix product, $\mathbf{S}_1(z)\mathbf{S}_0(z)$, followed by gains of $1/K$ and K. Analysis banks generally are given by products of alternating lower- and upper-triangular lifting matrices:

$$\mathbf{H}_a(z) = \mathrm{diag}(1/K,\, K)\, \mathbf{S}_{N_{LS}-1}(z) \cdots \mathbf{S}_1(z)\, \mathbf{S}_0(z)\ . \tag{43}$$

Note that this implies the normalization (14). The inverse of (43) is

$$\mathbf{H}_a^{-1}(z) \ = \ \mathbf{S}_0^{-1}(z) \cdots \mathbf{S}_{N_{LS}-1}^{-1}(z)\, \mathrm{diag}(K,\, 1/K)\ .$$

The inverses of lower- or upper-triangular lifting matrices are formed simply by negating the lifting filter. Since updates are added to both channels in the analysis bank of Figure 6, the corresponding synthesis bank is given by subtracting the same updates from the channels, as shown in Figure 7. The following useful relation shows that $\mathbf{L}$ algebraically intertwines $\mathbf{S}(z)$ and $\mathbf{S}^{-1}(z)$:

$$\mathbf{L}\,\mathbf{S}(z)\,\mathbf{L} = \mathbf{S}^{-1}(z)\ . \tag{44}$$

1) Gain conjugation: The following operation will be used in the derivation of linear phase lifting factorizations. Define the operation of conjugating a lifting matrix by a gain matrix,

$$\begin{aligned} C_K\mathbf{S}(z) \ &\equiv\ \mathrm{diag}(K,\, 1/K)\ \mathbf{S}(z)\ \mathrm{diag}(1/K,\, K) \qquad (45)\\ &= \begin{cases} \begin{bmatrix} 1 & K^2S(z) \\ 0 & 1 \end{bmatrix} & \text{if } \mathbf{S}(z) \text{ is upper-triangular,} \\[2ex] \begin{bmatrix} 1 & 0 \\ K^{-2}S(z) & 1 \end{bmatrix} & \text{if } \mathbf{S}(z) \text{ is lower-triangular.} \end{cases} \end{aligned}$$

This is equivalent to the intertwining relation

$$\mathbf{S}(z)\,\mathrm{diag}(1/K,\, K) = \mathrm{diag}(1/K,\, K)\, C_K\mathbf{S}(z)\ . \tag{46}$$

B. Reversible Implementation of Lifted Filter Banks

Lifting can be used to define invertible integer-to-integer transforms [14], [44], [45], [18]. Such transforms are referred to as *reversible filter banks* although, strictly speaking, such transforms are nonlinear and should be regarded as special nonlinear implementations of lifted (linear) filter banks. “Reversible” implies bit-perfect reconstruction in fixed-precision arithmetic; this is stronger than perfect reconstruction, which usually means “mathematically invertible in perfect arithmetic." Perfect reconstruction filter banks that are not reversible are called *irreversible.* Algebraically, a lifted filter bank is implemented reversibly by rounding each update term immediately before adding it to the channel being updated, as indicated in Figure 8 by the R operator. The synthesis bank in Figure 9 inverts this process losslessly by subtracting the same rounded updates. For example, the rounding rule used in JPEG 2000 is the floor function (rounding towards $-\infty$) following a bias offset, β (e.g., $\beta = 1/2$):

$$R(x) = \lfloor x + \beta \rfloor \,. \tag{47}$$

1) Dyadic lifting filters: To ensure rounding is robust with respect to implementation precision, the taps in the lifting filters of a reversible filter bank are usually restricted to dyadic rationals: rationals whose denominator is a power of two. Unlike rational numbers in general, dyadic rationals can be multiplied exactly in finite-precision arithmetic. For integer input, dyadic lifting filters output dyadic rationals, eliminating implementation-dependent ambiguities in the values going into the rounding operations. Since the output of a reversible analysis bank consists of integers, there are no output gains in Figure 8 or Figure 9. This implies that, for reversible filter banks, gain normalization [6, Annex G.2.1 and Annex H.1.1] must be built into the lifting filters. The use of dyadic rationals also reduces the complexity of implementing reversible filter banks since division by powers of two followed by rounding can be implemented using bit-shifts. Thus, only integer multiplications and bit-shifts are needed to implement reversible filter banks.

V. Lifting Structures for Whole-Sample Symmetric Filter Banks

We now derive the structure of the lifting steps for delay-minimized WS filter banks.

A. Lifting Filter Symmetries

Suppose $\mathbf{H}_a(z)$ is a WS analysis bank satisfying the delay-minimized symmetry convention, (28). Lift it to a new delay-minimized WS filter bank, $\mathbf{F}(z)$:

$$\mathbf{F}(z) = \mathbf{S}(z)\,\mathbf{H}_a(z) \,. \tag{48}$$

By Theorem 3, it follows that $\mathbf{S}(z)$ must also satisfy (28). Similarly, Theorem 3 also guarantees that applying a lifting matrix, $\mathbf{S}(z)$, satisfying (28) to a delay-minimized WS filter bank, $\mathbf{H}_a(z)$, will always yield a delay-minimized WS product, $\mathbf{S}(z)\,\mathbf{H}_a(z)$. The same conclusions hold for right-lifts, $\mathbf{F}(z) = \mathbf{H}_a(z)\,\mathbf{S}(z)$.

What do the lifting filters look like in lifting matrices that satisfy (28)? If $\mathbf{S}(z)$ is a lowpass (upper-triangular) lifting matrix, application of (28) implies that

$$S(z^{-1}) = zS(z) .$$

Similarly, if $\mathbf{S}(z)$ is a highpass (lower-triangular) lifting matrix, (28) implies

$$S(z^{-1}) = z^{-1}S(z) .$$

These results are summarized in

Lemma 8: A lifting matrix, $\mathbf{S}(z)$, lifts a WS filter bank satisfying (28) to another WS filter bank satisfying (28) if and only if $\mathbf{S}(z)$ also satisfies (28). An upper-triangular lifting matrix satisfies (28) if and only if its lifting filter is *half*-sample symmetric about $1/2$. A lower-triangular lifting matrix satisfies (28) if and only if its lifting filter is HS about $-1/2$.

It was noted in [19, Proposition 4] that these conditions on lifting filters are sufficient for preserving whole-sample symmetry when lifting a WS filter bank. It is important to distinguish between the *whole*-sample symmetric filter *bank* corresponding to the lifting matrix, $\mathbf{S}(z)$, and its *half*-sample symmetric lifting *filter*, $S(z)$. We will refer to a lifting matrix satisfying (28) as an *HS lifting step.*

B. Lifting Factorization

We now show that a WS filter bank can always be factored into a product of a lower-order WS filter bank and an HS lifting step. For simplicity, we consider a delay-minimized WS analysis bank, $\mathbf{H}_a(z)$, satisfying (14). Let the impulse responses, $h_0(n)$ and $h_1(n)$, be supported on the intervals $[-n_0,\ n_0]$ and $[-n_1 - 1,\ n_1 - 1]$, respectively, so that

$$\mathrm{order}(H_0) = 2n_0 \quad \text{and} \quad \mathrm{order}(H_1) = 2n_1 .$$

When at least one of the filters has order > 0, it was proven in [20, Formula (10)] that the sum of the lengths of the filters must be divisible by 4,

$$4 \mid 2n_0 + 2n_1 + 2 ,$$

which is equivalent to saying that $n_0 + n_1$ must be odd. In particular, one filter must be at least two terms longer than the other. A WS filter bank with two filters of order zero is a gain matrix, which is equivalent, modulo gain factors, to the lazy wavelet filter bank (30).

Any representation of a WS analysis filter bank as a cascade of HS lifting steps terminating with a diagonal gain matrix will be referred to as a *WS group lifting cascade.* The existence of such factorizations is presented in [3, Section 6.4.4] via the general process of universal lifting factorization and is based on factoring the inverse matrix, $\mathbf{H}_a^{-1}(z)$. Our treatment, which factors a WS analysis matrix directly in terms of explicitly specified HS lifting filters, is inspired by the Lifting Theorem [46], [47], [17], which characterizes all filters complementary to a given filter in terms of lifting operations. (Two filters are *complementary* if they form a perfect reconstruction analysis filter bank.)

The idea of the following theorem is to inductively reduce (or "downlift") a given WS analysis filter bank through a sequence of intermediate, lower-order complementary filter pairs by alternately factoring off lowpass and highpass HS lifting steps until both remaining filters have order zero. For instance, if $\mathrm{order}(H_0) > \mathrm{order}(H_1)$ the process looks like:

$$\{H_0, H_1\} \xrightarrow{\mathbf{S}_{N-1}^{-1}} \{H_0', H_1\} \xrightarrow{\mathbf{S}_{N-2}^{-1}} \{H_0', H_1'\} \xrightarrow{\mathbf{S}_{N-3}^{-1}} \{H_0'', H_1'\} \xrightarrow{\mathbf{S}_{N-4}^{-1}} \cdots \xrightarrow{\mathbf{S}_{0}^{-1}} \mathrm{diag}(1/K, K)\,. \tag{49}$$

By factoring off an HS lifting step each time, Theorem 3 ensures that each intermediate reduced-order filter bank is itself a WS filter bank. The proof is constructive and produces explicit formulas for the lifting filters arising in the factorization.

Theorem 9 (WS downlifting factorization): A WS filter bank, $\mathbf{H}_a(z)$, of determinant 1 satisfies the delay-minimized symmetry condition (28) if and only if it can be factored as a WS group lifting cascade,

$$\mathbf{H}_a(z) = \mathrm{diag}(1/K, K)\,\mathbf{S}_{N_{LS}-1}(z)\cdots\mathbf{S}_1(z)\mathbf{S}_0(z)\,, \tag{50}$$

where each lifting matrix, $\mathbf{S}_i(z)$, satisfies (28).

Proof: One direction is easy: Theorem 3 shows that cascades of HS lifting steps necessarily yield WS filter banks.

Conversely, suppose we are given a WS filter bank, $\mathbf{H}_a(z)$, satisfying (28). Assume that $n_0 > n_1$; i.e., $\mathrm{order}(H_0) > \mathrm{order}(H_1)$. (The proof for the other case is similar.)

Left-factor $\mathbf{H}_a(z)$ by a to-be-determined upper-triangular lifting matrix, $\mathbf{S}(z)$:

$$\mathbf{S}^{-1}(z)\mathbf{H}_a(z) = \begin{bmatrix} H_{a_{00}}(z) - S(z)H_{a_{10}}(z) & H_{a_{01}}(z) - S(z)H_{a_{11}}(z) \\ H_{a_{10}}(z) & H_{a_{11}}(z) \end{bmatrix}.$$

This is equivalent to

$$\begin{bmatrix} H_0'(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} H_0(z) - S(z^2)H_1(z) \\ H_1(z) \end{bmatrix}.$$

The Lifting Theorem says that $H_0'(z)$ is complementary to $H_1(z)$ for *any* choice of lifting filter, $S(z)$. To ensure that $H_0'(z)$ is WS and of strictly lower order than $H_0(z)$, define

$$S(z) \equiv c(z^{n_s-1} + z^{-n_s}) \quad \text{where} \tag{51}$$

$$c \equiv \frac{h_0(n_0)}{h_1(n_1-1)} \quad \text{and} \quad 2n_s \equiv n_0 - n_1 + 1 > 0\ .$$

$S(z)$ in formula (51) is HS about 1/2 so $\mathbf{S}(z)$ satisfies (28) by Lemma 8, and Theorem 3 implies that $\{H_0'(z), H_1(z)\}$ is WS. Moreover, since $H_0(z)$ is symmetric, c and n_s kill both the left- and right-outermost terms in $H_0(z)$: the support of the impulse response of $S(z^2)H_1(z)$ is

$$[-2(n_s-1) - n_1 - 1,\, 2n_s + n_1 - 1] = [-n_0,\, n_0]\ ,$$

and computing the outermost terms in $H_0'(z)$ gives

$$h_0'(n_0) = h_0(n_0) - ch_1(n_1-1) = 0\ .$$

Thus, factoring off the HS lowpass lifting step, $\mathbf{S}(z)$, leaves a new lowpass filter of strictly lower order. Since $n_0' + n_1$ must also be odd, it follows that factoring off $\mathbf{S}(z)$ actually reduces the order of $H_0(z)$ by a multiple of 4 if order$(H_0) > 3$. Indeed, one can use (14) to show that factorization actually kills the *two* outermost terms at each end of $H_0(z)$ since

$$\frac{h_0(n_0)}{h_1(n_1-1)} = \frac{h_0(n_0-1)}{h_1(n_1-2)}\ .$$

If order(H_0') > order(H_1) after downlifting by (51), one can factor off additional lowpass lifting steps until H_1 has a higher order than the remaining lowpass filter. Then use matrix multiplication to combine the successive lowpass lifts into a single upper-triangular matrix, $\mathbf{S}'(z)$, whose HS lifting filter is the sum of the HS lifting filters in the factor matrices.

Assuming the analogous order-reduction result for factoring off highpass HS lifting steps, $\mathbf{S}_i'(z)$, when order(H_0) < order(H_1), one can inductively reduce the higher-order filter at each step until both filters have order zero, resulting in a factorization of the form

$$\mathbf{H}_a(z) = \mathbf{S}'_{N_{LS}-1}(z) \cdots \mathbf{S}_1'(z)\mathbf{S}_0'(z)\,\mathrm{diag}(1/K, K)\ .$$

To move the gain matrix to the left end of the lifting cascade, use the intertwining relation (46). This establishes the factorization (50), with $\mathbf{S}_i(z) = C_K \mathbf{S}_i'(z)$. ■

Remark: JPEG 2000 Part 2 [6, Annex G] specifies WS filter banks using "folded" (nonredundant) representations of HS lifting filters, together with "folded" formulas for applying them.

VI. Lifting Structures for Half-Sample Symmetric Filter Banks

Now that we have considered in some detail the general structure theory for lifting factorizations of WS filter banks, we will attempt a similar approach for HS filter banks and see that the situation is more complicated in the HS case. We simplify the analysis by restricting the group delays of the filters to the concentric delay-minimized condition (37), though in this section we do not insist that the filter banks have determinant 1.

A. Lifting Filter Symmetries

Suppose that $\mathbf{H}_a(z)$ is an HS filter bank satisfying (37). Left-lift it to a new filter bank, $\mathbf{F}(z)$:

$$\mathbf{F}(z) = \mathbf{S}(z)\,\mathbf{H}_a(z)\ . \tag{52}$$

What condition on $\mathbf{S}(z)$ will force $\mathbf{F}(z)$ to satisfy (37)? In the HS case, there is no group to constrain the structure of the lifting matrix, so assume that $\mathbf{F}(z)$ also satisfies (37):

$$\begin{aligned} \mathbf{L}\,\mathbf{S}(z)\,\mathbf{H}_a(z)\,\mathbf{J} &= \mathbf{L}\,\mathbf{F}(z)\,\mathbf{J} \\ &= \mathbf{F}(z^{-1}) \\ &= \mathbf{S}(z^{-1})\,\mathbf{H}_a(z^{-1}) \\ &= \mathbf{S}(z^{-1})\,\mathbf{L}\,\mathbf{H}_a(z)\,\mathbf{J}\ . \end{aligned}$$

Simplification and (44) yield the following necessary condition on $\mathbf{S}(z)$:

$$\mathbf{S}(z^{-1}) \;=\; \mathbf{L}\,\mathbf{S}(z)\,\mathbf{L} \;=\; \mathbf{S}^{-1}(z)\ . \tag{53}$$

In both the lower- and upper-triangular cases, (53) says that the lifting filter satisfies

$$S(z^{-1}) = -S(z)\ ,$$

i.e., it is whole-sample *anti*-symmetric (WA) about $n = 0$. Similar analysis shows that (53) is also a sufficient condition for lifting an HS filter bank to another HS filter bank. In summary:

Lemma 10: If either $\mathbf{H}_a(z)$ or $\mathbf{F}(z)$ in (52) is an HS filter bank satisfying the concentric delay-minimized condition (37), then the other filter bank also satisfies (37) if and only if $\mathbf{S}(z)$ satisfies (53), which is equivalent to having a lifting filter that is WA about 0.

It was noted in [19, Proposition 4] that this condition on lifting filters is sufficient for preserving half-sample symmetry when lifting an HS filter bank. In analogy with Lemma 8, we refer to lifting matrices satisfying (53) as *WA lifting steps.* Unlike the WS case, however, WA lifting steps are not HS polyphase matrices in their own right; i.e., a WA lifting matrix, $\mathbf{S}(z)$, never satisfies (37). In fact, it is easy to show the following:

Lemma 11: For any lifting matrix, $\mathbf{S}(z)$, the matrix $\mathbf{L}\,\mathbf{S}(z)\,\mathbf{J}$ never has the algebraic form of either a lower- or upper-triangular lifting matrix.

In contrast to the WS case, right-lifting doesn't work with HS filter banks:

Theorem 12 (No right-lifting for HS filter banks): Suppose that $\mathbf{H}_a(z)$ is an HS filter bank satisfying the concentric delay-minimized condition (37). If $\mathbf{F}(z)$ is right-lifted from $\mathbf{H}_a(z)$,

$$\mathbf{F}(z) = \mathbf{H}_a(z)\,\mathbf{S}(z)\ ,$$

then $\mathbf{F}(z)$ can only be a concentric delay-minimized HS filter bank if $\mathbf{S}(z) = \mathbf{I}$ and $\mathbf{F}(z) = \mathbf{H}_a(z)$.

Proof: Suppose $\mathbf{F}(z)$ is a concentric delay-minimized HS filter bank satisfying (37):

$$\begin{aligned}
\mathbf{L}\,\mathbf{H}_a(z)\,\mathbf{S}(z)\,\mathbf{J} &= \mathbf{L}\,\mathbf{F}(z)\,\mathbf{J} \\
&= \mathbf{H}_a(z^{-1})\,\mathbf{S}(z^{-1}) \\
&= \mathbf{L}\,\mathbf{H}_a(z)\,\mathbf{J}\,\mathbf{S}(z^{-1})\ .
\end{aligned}$$

Simplification yields the following necessary condition on $\mathbf{S}(z)$:

$$\mathbf{S}(z^{-1}) = \mathbf{J}\,\mathbf{S}(z)\,\mathbf{J}\,. \tag{54}$$

Conjugation by $\mathbf{J}$ is a double-transpose, so (54) says that $\mathbf{S}(z^{-1})$ has the same lifting filter as $\mathbf{S}(z)$ but the *opposite* update characteristic, which implies $\mathbf{S}(z) = \mathbf{I}$. ■

Also unlike the WS case, there are no trivial (zeroth order) HS filter banks. Any HS filter bank with two first-order filters is equivalent, modulo gain factors, to the Haar filter bank (39). This immediately presents a factorization problem: one can verify that the Haar filter bank cannot be lifted from the identity using WA lifting steps. Indeed, the lifting factorization of the Haar filter bank in JPEG 2000 Part 2 [6, Annex H.4.1.1.1] uses *delay-free* lifting steps,

$$\mathbf{S}_0(z) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\ ,\quad \mathbf{S}_1(z) = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}. \tag{55}$$

B. Lifting Factorization

Consider an HS filter bank, $\mathbf{H}(z)$, satisfying the concentric delay-minimized condition (37), and try to factor it into a lower-order HS filter bank and a WA lifting step. The impulse responses, h_0 and h_1, are supported on the intervals $[-n_0-1,\, n_0]$ and $[-n_1-1,\, n_1]$, so that

$$\mathrm{order}(H_0) = 2n_0 + 1 \quad \text{and} \quad \mathrm{order}(H_1) = 2n_1 + 1\ .$$

It was proven in [20, Formula (14)] that the sum of the filter lengths must be divisible by 4,

$$4 \mid 2n_0 + 2n_1 + 4 ,$$

which is equivalent to saying that $n_0 + n_1$ must be *even.*

Assume that $\mathrm{order}(H_0) > \mathrm{order}(H_1)$ and follow the downlifting scheme in (49):

$$H_0'(z) = H_0(z) - S(z^2)H_1(z) . \tag{56}$$

By Lemma 10 we know that $S(z)$ must be WA if $H_0'(z)$ is to be HS, so define

$$S(z) \equiv c(-z^{n_s} + z^{-n_s}) \quad \text{where} \tag{57}$$
$$c \equiv \frac{h_0(n_0)}{h_1(n_1)} \quad \text{and} \quad 2n_s \equiv n_0 - n_1 > 0 .$$

As in Theorem 9, this downlifting step kills the two outermost terms at each end of $H_0(z)$, and additional WA lowpass lifting steps can be factored off if $\mathrm{order}(H_0') > \mathrm{order}(H_1)$. Similar WA highpass lifting step factorizations are possible when $\mathrm{order}(H_0) < \mathrm{order}(H_1)$.

Now we confront a problem that didn't occur in the WS case: what happens if the filters in a concentric HS filter bank have *equal* orders? If $H_0(z)$ and $H_1(z)$ are concentric, equal-length filters, it follows that any WA lowpass lift (56) necessarily *increases* the order of $H_0(z)$. The only way to reduce order in an equal-length HS filter bank is by factoring off a zeroth-order lifting step. Since such a step cannot be WA, Lemma 10 implies that the downlifted filter bank will not be HS. Note that this is exactly what happens in the standard factorization of the Haar filter bank (55). Higher-order HS filter banks can be lifted from the Haar filter bank using WA lifting steps, but the lifting factorization of such filter banks will always include non-WA steps like those in (55). For instance, the 2-tap/10-tap HS filter bank specified in Part 2 [6, Annex H.4.1.1.3] is lifted from the Haar via a highpass 4th-order WA lifting update.

Another important example of this obstruction to WA downlifting is the 6-tap/10-tap HS filter bank specified in [6, Annex H.4.1.2.1]. This filter bank was originally constructed by spectral factorization techniques, and it has a lifting factorization of the form

$$\mathbf{H}_a(z) = \mathbf{S}(z)\mathbf{B}(z) ,$$

where $S(z)$ is a second-order WA filter and $\mathbf{B}(z)$ is an equal-length (6-tap/6-tap) concentric HS filter bank. As stated above, $\mathbf{B}(z)$ cannot be factored further using WA lifting steps; a lifting factorization of $\mathbf{B}(z)$ in terms of non-WA lifting steps is given in [6, Annex H.4.1.2.1], and other lifting factorizations are possible using the general procedures described in [17].

We refer to $\mathbf{B}(z)$ as an *equal-length HS base filter bank*. Since it is always possible to factor off a WA lifting step from an HS filter bank with *unequal* length filters, it is natural to ask whether it is possible, through clever factorization, to avoid equal-length base HS filter banks when factoring an HS filter bank into WA lifting steps. Unfortunately, this is never possible:

Theorem 13: A concentric delay-minimized HS filter bank (not necessarily of determinant 1) can never be factored completely into a cascade of WA lifting steps.

Proof: Suppose it can,

$$\mathbf{H}_a(z) = \mathrm{diag}(K_0, K_1)\, \mathbf{S}_N(z) \cdots \mathbf{S}_0(z)\ ,$$

where each lifting step satisfies (53). This can be written

$$\mathbf{S}_0^{-1}(z) \cdots \mathbf{S}_N^{-1}(z)\, \mathrm{diag}(1/K_0, 1/K_1)\, \mathbf{H}_a(z) = \mathbf{I}\ .$$

By Lemma 10, it follows that $\mathbf{I}$ is a concentric delay-minimized HS filter bank, a contradiction. Therefore, a complete factorization of $\mathbf{H}_a(z)$ into WA lifting steps cannot exist. ∎

We summarize factorization of HS filter banks in

Theorem 14 (HS downlifting factorization): $\mathbf{H}_a(z)$ satisfies the concentric delay-minimized HS convention (37) if and only if it can be factored into a cascade of alternating lowpass and highpass WA lifting steps satisfying (53) and an equal-length HS base satisfying (37):

$$\mathbf{H}_a(z) = \mathbf{S}_{N_{LS}-1}(z) \cdots \mathbf{B}(z)\ . \tag{58}$$

1) Preservation of subband symmetries generated by HS filter banks: We now prove that WA lifting steps preserve the symmetries of HS filter bank output subbands. Suppose that $\mathbf{Y}(z)$ is a polyphase vector that has the symmetry described by (40); e.g., it could consist of the output generated by an equal-length HS base filter bank. In analogy with Theorem 4, the following result shows that subsequent WA lifting steps preserve the symmetries of y_0 and y_1 by showing that WA lifting steps preserve equation (40).

Theorem 15: Let $\mathbf{Y}(z)$ satisfy (40), and let $\mathbf{S}(z)$ be a WA lifting step, either lower- or upper-triangular. If $\mathbf{Y}'(z) = \mathbf{S}(z)\mathbf{Y}(z)$ then $\mathbf{Y}'(z)$ also satisfies (40).

Proof: Proceeding as in the proof of Theorem 4,

$$\begin{aligned}
\mathbf{Y}'(z^{-1}) &= \mathbf{S}(z^{-1})\, \mathbf{Y}(z^{-1}) \\
&= \mathbf{S}^{-1}(z)\, \mathbf{Y}(z^{-1}) \quad \text{by (53)} \\
&= \mathbf{S}^{-1}(z)\, z^{(2i_0-1)/2}\, \mathbf{L}\mathbf{Y}(z) \quad \text{by (40)}
\end{aligned}$$

$$\begin{aligned} &= z^{(2i_0-1)/2}\,\mathbf{L}\,\mathbf{S}(z)\,\mathbf{Y}(z) \quad \text{by (44)} \\ &= z^{(2i_0-1)/2}\,\mathbf{L}\,\mathbf{Y}'(z)\;. \end{aligned}$$

This is formula (40), so $\mathbf{Y}'(z)$ has the same symmetries as $\mathbf{Y}(z)$. ∎

2) Arbitrary filter banks in JPEG 2000: Because of the problems associated with equal-length HS base filter banks, Part 2 Annex H contains extensions that accommodate not only HS filter banks but also completely arbitrary FIR filter banks. This is done by allowing arbitrary (e.g., asymmetric) FIR lifting filters with arbitrary group delays. By the universality of lifting [17], this allows *any* two-channel FIR filter bank to be signaled in a JPEG 2000 codestream. Allowing completely arbitrary FIR lifting factorizations eliminates any need to rely on a particular factorization structure for equal-length HS base filter banks. This also allows users to employ nonlinear-phase filter banks, such as paraunitary (orthogonal) filter banks, which may have advantages in certain applications.

VII. Conclusions

This paper has developed the matrix theory for the "polyphase-with-advance" representation that underlies the theory of lifting for two-channel perfect reconstruction filter banks. Our approach has emphasized the simplifying role played by the algebraic groups that arise naturally in the polyphase structures associated with linear phase filter banks (Theorem 1 and Theorem 3). We have obtained new, rigorous proofs of such results as the characterization of the effects of a WS filter bank (or HS lifting steps, which are WS filter banks in their own right) on the symmetry of a whole-sample symmetric input signal (Theorem 4), and the existence of HS lifting step factorizations for WS filter banks (Theorem 9). These results have been generalized using the polyphase-with-advance theory to yield new theorems about the effects of an HS filter bank and its WA lifting steps on the symmetry of a half-sample symmetric input signal (Theorem 7 and Theorem 15). We have also proven the existence of WA lifting step factorizations for HS filter banks (Theorem 14) and have shown that equal-length HS base filter banks, which *cannot* be factored into WA lifting steps, are unavoidable when factoring HS filter banks (Theorem 13). The theory has been applied to developing the foundations for the specification of filter banks via lifting in Part 2 of the ISO/IEC JPEG 2000 image coding standard.

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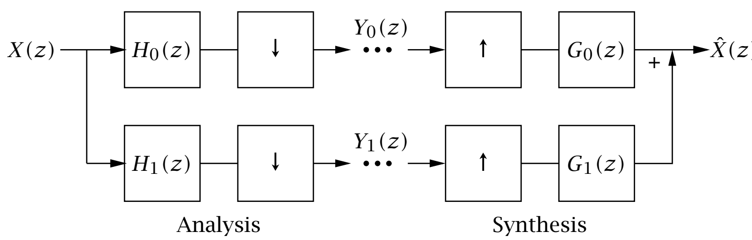

Fig. 1. Direct-form representation of a two-channel multirate filter bank.

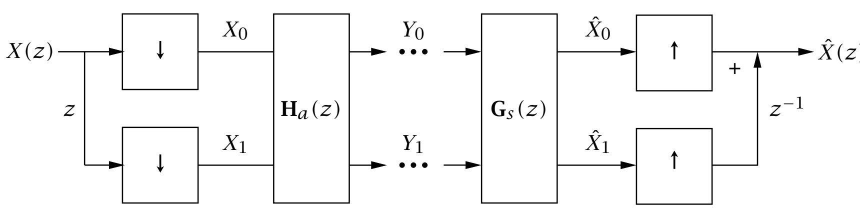

Fig. 2. The polyphase-with-advance filter bank representation.

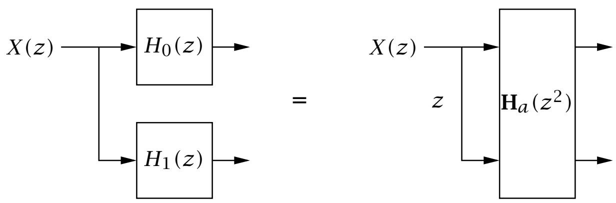

Fig. 3. Relation between direct-form and polyphase-with-advance analysis filtering.

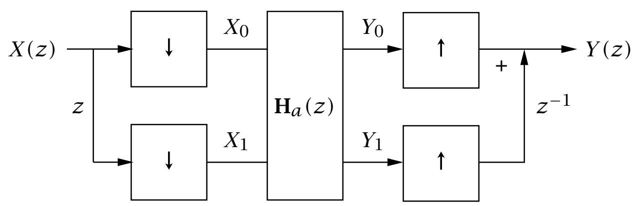

Fig. 4. Analysis filtering with interleaved output.

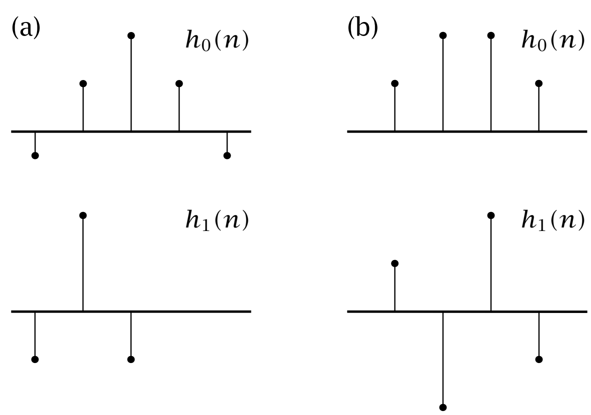

Fig. 5. (a) Whole-sample symmetric filter bank. (b) Half-sample symmetric filter bank.

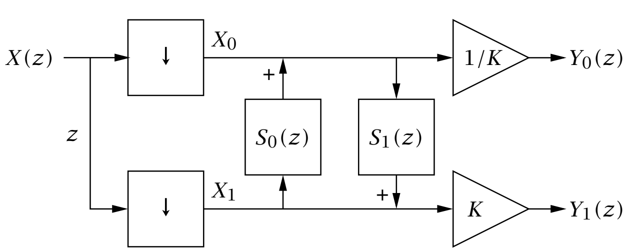

Fig. 6. Lifting form of a two channel irreversible analysis filter bank.

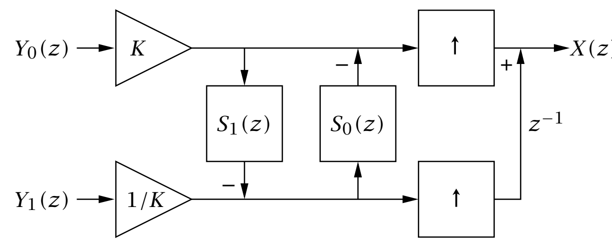

Fig. 7. Lifting form of a two channel irreversible synthesis filter bank.

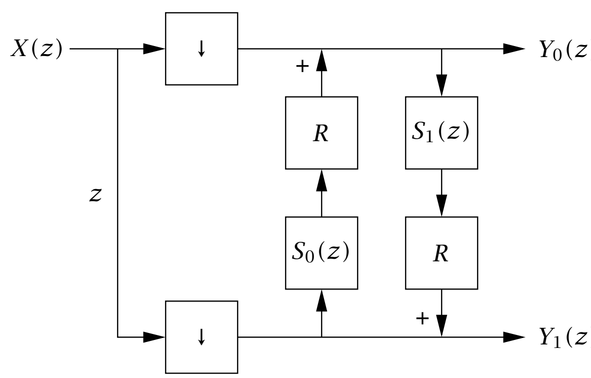

Fig. 8. Reversible implementation of a lifted analysis filter bank.

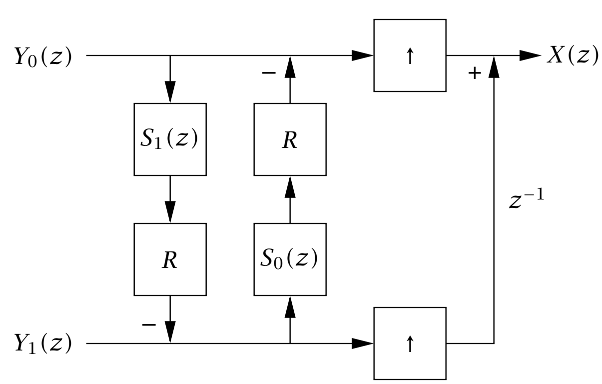

Fig. 9. Reversible implementation of a lifted synthesis filter bank.