LIFTED LINEAR PHASE FILTER BANKS AND THE POLYPHASE-WITH-ADVANCE REPRESENTATION

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ABSTRACT

We develop the noncausal polyphase-with-advance representation that underlies the theory of lifted filter banks and wavelet transforms used in the ISO/IEC JPEG 2000 image coding standard. The theory includes a matrix algebra framework for analyzing linear phase two-channel filter banks. Whole- and half-sample symmetric filter banks are characterized completely in terms of the theory, and linear phase lifting factorizations are developed for both classes of linear phase filter banks.

1. INTRODUCTION

The polyphase-with-advance representation is a noncausal structure for two-channel perfect reconstruction multirate filter banks [1, 2, 3, 4]. The polyphase-with-advance representation underlies the theory of lifted filter banks and wavelet transforms as developed by Sweldens and Daubechies [5, 6] and provides the setting for the lifting factorizations used in the ISO/IEC JPEG 2000 still image coding standard [7, 8, 9], which the authors helped to develop.

Our treatment [10] of the polyphase-with-advance representation develops an extensive matrix algebra framework that goes far beyond the results of [6]. Specifically, we focus on analyzing and implementing linear phase two-channel filter banks via linear phase lifting cascade schemes. Whole-sample symmetric (WS) and half-sample symmetric (HS) linear phase filter banks, shown in Figure 1, are characterized completely in terms of the polyphase-with-advance representation. The theory benefits significantly from several group-theoretic structures arising in the polyphase-with-advance representation.

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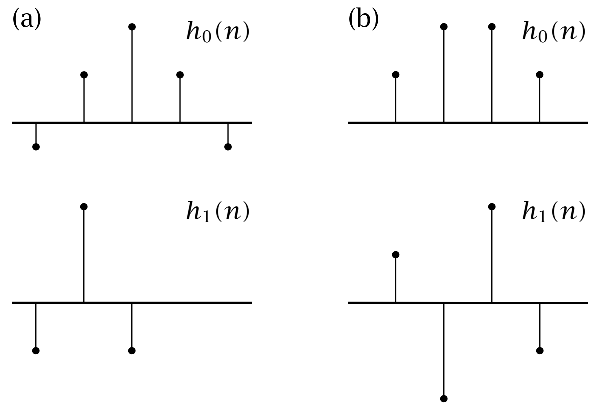


Fig. 1. (a) Whole-sample symmetric filter bank. (b) Half-sample symmetric filter bank.

It is known that the polyphase matrices of perfect reconstruction FIR filter banks form an infinite-dimensional nonabelian group, but it seems to be much less widely recognized that WS polyphase matrices form a subgroup of the FIR filter bank group. Although WS filter banks are treated in [9], which provides extensive coverage of Part 1 (the “Baseline”) of the JPEG 2000 standard, we simplify the study of WS filter banks using these elementary group-theoretic concepts. Our approach also simplifies matters by using the polyphase-with-advance representation to avoid the time-varying convolution operation defined in [9].

More generally, however, our goal is to provide a thorough treatment of the general filter bank theory behind the Part 2 algorithms for user-defined wavelet transforms, which has not been done to date in the published literature. As part of this treatment [11], we derive one of the more mysterious filter bank specifications in the JPEG 2000 standard: the requirements for user-defined filter bank normalization. Other work in preparation based on the polyphase-with-advance theory [12] analyzes the failure of lossless reconstruction for reversible HS filter banks in the context of symmetric boundary extension techniques [13, 14].

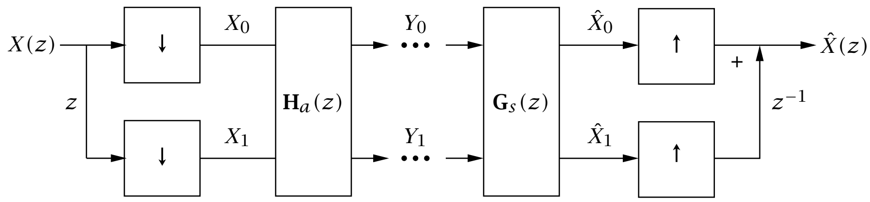


Fig. 2. The polyphase-with-advance filter bank representation.

2. THE POLYPHASE-WITH-ADVANCE REPRESENTATION

Lifting is defined in JPEG 2000 by low-level arithmetic expressions derived from the development presented in [6], which is based on the polyphase representation shown in Figure 2. This particular style of polyphase representation is called "polyphase with advance" in [3] because of the one-sample advance used to demultiplex the input into even- and odd-indexed samples in the analysis bank.

The analysis polyphase matrix, $\mathbf{H}_a(z)$, acts on the vector of even and odd channels,

$$\mathbf{X}(z) = \begin{bmatrix} X_0(z) \\ X_1(z) \end{bmatrix} . \quad (1)$$

It is shown in [10] that whole-sample symmetry of x about $i_0 \in \mathbb{Z}$ can be expressed equivalently in terms of the transform-domain polyphase vector as

$$\mathbf{X}(z^{-1}) = z^{i_0}\boldsymbol{\Lambda}(z)\mathbf{X}(z) \quad \text{where} \quad (2)$$
$$\boldsymbol{\Lambda}(z) \equiv \mathrm{diag}(1, z^{-1}) .$$

Similarly, it is shown that $\mathbf{H}_a(z)$ is the polyphase matrix of a WS filter bank whose lowpass and highpass filters have group delays $\gamma_0 = 0$ and $\gamma_1 = -1$, respectively, if and only if

$$\mathbf{H}_a(z^{-1}) = \boldsymbol{\Lambda}(z)\,\mathbf{H}_a(z)\,\boldsymbol{\Lambda}(z^{-1}) , \quad (3)$$

which implies

$$\mathbf{G}_s(z^{-1}) = \boldsymbol{\Lambda}(z)\,\mathbf{G}_s(z)\,\boldsymbol{\Lambda}(z^{-1}) . \quad (4)$$

Together, (3) and (4) imply

Theorem 1 (WS Group Property) *WS filter banks satisfying (3) form a subgroup (the* WS subgroup*) of the FIR filter bank group.*

This fact simplifies a number of details involved in factoring or synthesizing WS filter banks in the polyphase domain.

Another advantage of the polyphase-with-advance matrix algebra is that it simplifies the analysis of symmetry properties for filter bank output subbands in interleaved form, the description of filter bank output employed in the JPEG 2000 standard. The following result is used in [9] to perform symmetric extension transforms [13, 14] on finite-length input signals using an equivalent extension policy in the lifting domain. The interested reader may compare the short proof from [10] using polyphase-with-advance matrix algebra with the proof in [9] using a special form of time-varying convolution.

Theorem 2 *Let $x(n)$ be whole-sample symmetric about i_0, and let $y(n)$ be the interleaved output from an analysis filter bank, $\mathbf{H}_a(z)$. If $\mathbf{H}_a(z)$ is WS with group delays $\gamma_0 = 0$ and $\gamma_1 = -1$ then the interleaved output is also whole-sample symmetric about i_0.*

The analogous theory for HS filter banks leads to quite different results. Unlike (3), which says that the polyphase components of WS filters are symmetric, the polyphase components of HS filters are mirror images of each other. When both filters share the same axis of symmetry, $\gamma_0 = -1/2 = \gamma_1$, this is expressed algebraically in [10] as

$$\mathbf{H}_a(z^{-1}) = \mathbf{L}\,\mathbf{H}_a(z)\,\mathbf{J} \quad \text{where} \quad (5)$$

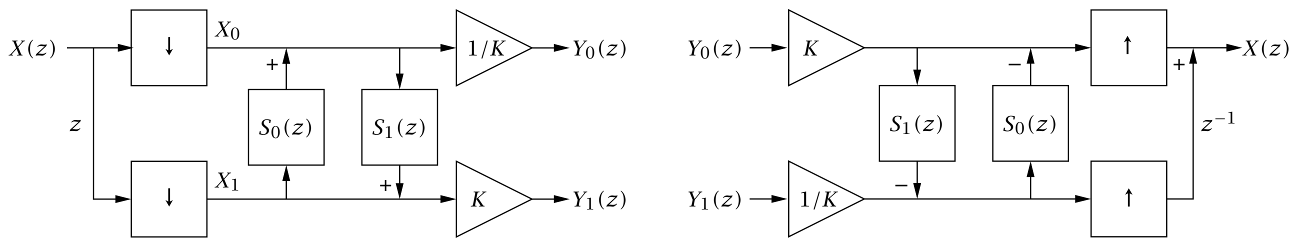
$$\mathbf{L} \equiv \mathrm{diag}(1, -1) \quad ; \quad \mathbf{J} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \quad (6)$$

Unlike WS filter banks, these relationships do *not* define a group. While there is an analog of Theorem 2 for HS filter banks, developing extension policies in the lifting domain that are mathematically equivalent to symmetric extension for HS filter banks turns out to be greatly complicated by difficulties arising in the lifting factorization of HS filter banks into linear phase lifting steps.

3. LIFTING DECOMPOSITIONS

A lifting representation for a polyphase-with-advance analysis matrix is defined to be a cascade-form decomposition,

$$\mathbf{H}_a(z) = \mathrm{diag}(1/K, K)\mathbf{S}_{N_{LS}-1}(z)\cdots\mathbf{S}_1(z)\mathbf{S}_0(z) , \quad (7)$$



(a) Analysis bank.

(b) Synthesis bank.

Fig. 3. Lifting factorization in the polyphase-with-advance representation.

where the factor matrices, $\mathbf{S}_i(z)$, are alternately lower- or upper-triangular with 1's on the diagonal. For instance, the lowpass (even-channel) update in Figure 3(a) is described by an upper-triangular matrix,

$$\mathbf{S}_0(z) = \begin{bmatrix} 1 & S_0(z) \\ 0 & 1 \end{bmatrix}, \tag{8}$$

while the highpass (odd-channel) update is described by a lower-triangular matrix,

$$\mathbf{S}_1(z) = \begin{bmatrix} 1 & 0 \\ S_1(z) & 1 \end{bmatrix}. \tag{9}$$

The synthesis bank, $\mathbf{G}_s(z) = \mathbf{H}_a^{-1}(z)$, is

$$\mathbf{G}_s(z) = \mathbf{S}_0^{-1}(z) \cdots \mathbf{S}_{N_{LS}-1}^{-1}(z)\,\mathrm{diag}(K, 1/K), \tag{10}$$

where the inverses of lifting matrices are formed simply by negating the lifting filter. For instance, the inverse of the lowpass update matrix in (8) is

$$\mathbf{S}_0^{-1}(z) = \begin{bmatrix} 1 & -S_0(z) \\ 0 & 1 \end{bmatrix}. \tag{11}$$

Figure 3(b) depicts the lifted synthesis bank corresponding to the analysis bank in Figure 3(a).

3.1. Lifting Structures for WS Filter Banks

Suppose $\mathbf{H}_a(z)$ is a WS analysis bank satisfying (3) and left-lift it to a new filter bank, $\mathbf{F}(z)$:

$$\mathbf{F}(z) = \mathbf{S}(z)\,\mathbf{H}_a(z). \tag{12}$$

By Theorem 1, it follows immediately that $\mathbf{F}(z)$ is WS if and only if $\mathbf{S}(z)$ also satisfies (3). The same conclusion holds for right-lifts, $\mathbf{F}(z) = \mathbf{H}_a(z)\,\mathbf{S}(z)$.

The lifting filters in lifting matrices satisfying (3) are characterized in [10] by

Lemma 3 *A lifting matrix, $\mathbf{S}(z)$, lifts a WS filter bank to another WS filter bank if and only if $\mathbf{S}(z)$ also satisfies (3). An upper-triangular lifting matrix satisfies (3) if and only if its lifting filter is* half-*sample symmetric about* $1/2$*. A lower-triangular lifting matrix satisfies (3) if and only if its filter is HS about* $-1/2$*.*

It is important to distinguish between the *whole*-sample symmetric filter *bank* corresponding to the lifting matrix, $\mathbf{S}(z)$, and its *half*-sample symmetric lifting *filter*, $S(z)$. We refer to a lifting matrix satisfying (3) as an *HS lifting step.*

Once we know that the only way to lift one WS filter bank to another is using HS lifting filters, it is natural to ask whether it is always possible to *factor* a WS filter bank completely into HS lifting steps. The existence of such factorizations was proven in [9], and in [10] we present an explicit construction of the factors via an inductive order-reducing factorization process, inspired by the Lifting Theorem [15, 16].

Theorem 4 $\mathbf{H}_a(z)$ *is a WS filter bank if and only if it can be factored as a WS group product,*

$$\mathbf{H}_a(z) = \mathit{diag}(1/K, K)\,\mathbf{S}_{N_{LS}-1}(z) \cdots \mathbf{S}_0(z), \tag{13}$$

where each lifting filter, $S_i(z)$, is half-sample symmetric.

3.2. Lifting Structures for HS Filter Banks

Now suppose that $\mathbf{H}_a(z)$ is an HS filter bank satisfying (5) and left-lift it to a new filter bank, $\mathbf{F}(z)$, as in (12). In [10] we show

Lemma 5 $\mathbf{F}(z)$ *also satisfies (5) if and only if* $\mathbf{S}(z)$ *satisfies*

$$\mathbf{S}(z^{-1}) = \mathbf{L}\,\mathbf{S}(z)\,\mathbf{L} = \mathbf{S}^{-1}(z), \tag{14}$$

which is equivalent to having a lifting filter that is whole-sample anti-*symmetric (WA) about* $n = 0$.

Because there is no group law for HS filter banks, the situation for right-lifts is totally different:

Theorem 6 (No right-lifting HS filter banks) *Suppose* $\mathbf{H}_a(z)$ *is an HS filter bank. If* $\mathbf{F}(z)$ *is right-lifted from* $\mathbf{H}_a(z)$,

$$\mathbf{F}(z) = \mathbf{H}_a(z)\,\mathbf{S}(z),$$

then $\mathbf{F}(z)$ *can only be an HS filter bank if* $\mathbf{S}(z) = \mathbf{I}$ *and* $\mathbf{F}(z) = \mathbf{H}_a(z)$.

Also unlike the WS case, there are no trivial (zeroth order) HS filter banks. Any HS filter bank with two first-order filters is equivalent, modulo gain factors, to the Haar filter bank. This immediately presents a factorization problem: one can verify by direct calculation that the Haar filter bank cannot be lifted from a diagonal gain matrix, $\mathrm{diag}(1/K, K)$, using WA lifting steps. Indeed, the lifting factorization of the Haar filter bank specified in JPEG-2000 Part 2 [8, Annex H.4.1.1.1] uses *delay-free* lifting steps,

$$\mathbf{S}_0(z) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} , \ \mathbf{S}_1(z) = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} . \quad (15)$$

This behavior is actually typical of HS filter banks whose filters have *equal* orders. If we attempt to factor off a WA lifting step, it follows that any such factorization necessarily *increases* the order of the filter bank. The only way to reduce order in an equal-length HS filter bank is by factoring off a zeroth-order lifting step. Since such a step cannot be WA, Lemma 5 implies that the downlifted filter bank will not be HS, which is exactly what happens in the factorization of the Haar filter bank (15). Higher-order HS filter banks can be lifted from the Haar filter bank using WA lifting steps, but the lifting factorization of such filter banks will always include non-WA steps like those in (15). For instance, the 2-tap/10-tap HS filter bank specified in Part 2 [8, Annex H.4.1.1.3] is lifted from the Haar filter bank via a highpass 4th-order WA lifting update.

Another important example of this obstruction to WA factorization is the 6-tap/10-tap HS filter bank specified in [8, Annex H.4.1.2.1]. This filter bank was originally constructed by spectral factorization techniques, and it has a lifting factorization of the form

$$\mathbf{H}_a(z) = \mathbf{S}(z)\mathbf{B}(z) ,$$

where $S(z)$ is a second-order WA filter and $\mathbf{B}(z)$ is an equal-length (6-tap/6-tap) HS filter bank. As above, $\mathbf{B}(z)$ cannot be factored further using WA lifting steps. We refer to $\mathbf{B}(z)$ as an *equal-length HS base filter bank.* Such base filter banks are unavoidable when factoring HS filter banks:

Theorem 7 *$\mathbf{H}_a(z)$ is an HS filter bank if and only if it can be factored as*

$$\mathbf{H}_a(z) = diag(1/K, K)\, \mathbf{S}_{N_{LS}-1}(z) \cdots \mathbf{B}(z) . \quad (16)$$

Each lifting filter, $S_i(z)$ is WA, and $\mathbf{B}(z)$ is an equal-length base HS filter bank.

4. NORMALIZATION OF FILTER BANKS IN JPEG 2000

In [11] we derive one of the more mysterious specifications in the JPEG 2000 standard: the requirements for filter bank normalization. Let $\mathbf{E}(z)$ denotes the unnormalized cascade of lifting steps in (7),

$$\mathbf{E}(z) \equiv \mathbf{S}_{N_{LS}-1}(z) \cdots \mathbf{S}_1(z)\mathbf{S}_0(z) .$$

Given a lifted analysis filter bank,

$$\mathbf{H}_a(z) = \mathrm{diag}(1/K, K)\, \mathbf{E}(z) ,$$

the unstated intention of the normalization specification in JPEG 2000 Part 2 is to set the constant K so that the analysis bank will have unit lowpass gain at DC:

$$H_0(1) = 1 . \quad (17)$$

To achieve this, K must equal the DC gain of the *unnormalized* lowpass filter in $\mathbf{E}(z)$,

$$K = E_0(1) \neq 0 . \quad (18)$$

Let $\mathbf{E}^{(n)}(z)$ denote the n^{th} partial product of lifting steps in an analysis filter bank:

$$\mathbf{E}^{(n)}(z) = \mathbf{S}_n(z) \cdots \mathbf{S}_0(z) \text{ for } n = 0, \ldots, N_{LS} - 1.$$

In [11] it is shown that the vector of corresponding time-domain filters is:

$$\begin{aligned} \underline{E}^{(n)}(z) &\equiv \begin{bmatrix} E_0^{(n)}(z) \\ E_1^{(n)}(z) \end{bmatrix} \\ &= \mathbf{S}_n(z^2) \cdots \mathbf{S}_0(z^2) \begin{bmatrix} 1 \\ z \end{bmatrix} . \end{aligned} \quad (19)$$

Using (19), it follows that the vector of DC gains can be built up from the lifting steps via the recursion:

$$\underline{E}^{(n)}(1) = \mathbf{S}_n(1)\, \underline{E}^{(n-1)}(1) \text{ for } n = 0, \ldots, N_{LS} - 1, \quad (20)$$

where

$$\underline{E}^{(-1)}(1) \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

The goal is computing $K \equiv E_0^{(N_{LS}-1)}(1)$ via a scalar version of (20). We show in [11] that this is accomplished in JPEG 2000 Part 2 [8, Annex G.2.1] via the mathematically equivalent scalar recursion

$$B_n = D_n B_{n-1} + B_{n-2} \text{ for } n = 0, \ldots, N_{LS} - 1 , \quad (21)$$

where

$$B_{-1} \equiv B_{-2} \equiv 1 , \ D_n \equiv S_n(1) .$$

For irreversible filter banks, the constant, K, is

$$K = B_{N_{LS}-1} \text{ if } m_{init} = 0\,; \quad K = B_{N_{LS}-2} \text{ if } m_{init} = 1\,.$$

The flag m_{init} indicates whether the last analysis lifting step is a lowpass update ($m_{init} = 0$) or a highpass update ($m_{init} = 1$).

Finally, if the filter bank is *reversible* then there are no gain factors in either the analysis or synthesis filter banks. In this case, the standard requires that the lifting filters be scaled so that the *unnormalized* lowpass DC gain is unity. This is expressed in [8, Annex G.2.1.1] for reversible filter banks by

$$B_{N_{LS}-1} = 1 \text{ if } m_{init} = 0\,; \quad B_{N_{LS}-2} = 1 \text{ if } m_{init} = 1\,.$$

5. CONCLUSIONS

This paper has described the matrix theory for the "polyphase-with-advance" representation that underlies the theory of lifting for two-channel perfect reconstruction filter banks. Our approach has emphasized the simplifying role played by the algebraic groups that arise naturally in the polyphase structures associated with linear phase filter banks. The matrix theory has been applied to developing the foundations for the specification of filter banks via lifting in Part 2 of the ISO/IEC JPEG-2000 image coding standard. We have obtained new, rigorous proofs of such results as the characterization of the effects of a WS filter bank (or HS lifting steps, which are WS filter banks in their own right) on the symmetry of a whole-sample symmetric input signal, and the existence of HS lifting step factorizations for WS filter banks. These results have been generalized using the polyphase-with-advance theory to prove the existence of WA lifting step factorizations for HS filter banks. We have also shown that equal-length HS base filter banks, which *cannot* be factored into WA lifting steps, are unavoidable when factoring HS filter banks. Finally, we have outlined the normalization specifications for user-defined filter banks in Part 2 of the JPEG 2000 standard.

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